แบบจำลองคณิตศาสตร์ของผลเฉลยคลื่นเคลื่อนที่ในเครือข่ายประสาท

Mathematical Model of Traveling Wave Front Solutions in Neural Network

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บทคัดย่อ

ในงานวิจัยนี้ เราศึกษาถึงลักษณะรูปร่างของผลเฉลยคลื่นเคลื่อนที่สำหรับเซลประสาทสองชั้นที่มีลักษณะต่อเนื่องซึ่งเซลมีการยับยั้งภายในชั้นที่หนึ่ง เซลชั้นที่หนึ่งยับยั้งเซลชั้นที่สอง และมีการส่งผลกระตุ้นกลับสู่เซลชั้นที่หนึ่ง เราได้ค้นพบคุณลักษณะพฤติกรรมของคลื่นซึ่งขึ้นอยู่กับการเปลี่ยนแปลงในระดับการกระตุ้นต่ำสุดที่ทำให้เกิดการตอบสนองและระดับชั้นของมาตราส่วนเวลาระหว่างสองชั้นของเซลประสาท

คำสำคัญ : ผลเฉลยของคลื่นเคลื่อนที่ แบบจำลองทางคณิตศาสตร์ เครือข่ายประสาท

ABSTRACT

In this research, we will study the shape of a traveling wave front solution for a double continuous layers of nerve cells was considered, involving a mutually inhibitory layer of cells coupled via inhibitory connections to a second layer of cells that provides excitatory feedback connections to the first layer. Finally, results on the qualitative behavior of wave fronts that depend on the threshold parameter and the time scales parameter were derived.

Key Words : Traveling wave front solutions, Mathematical modeling, Neural network

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INTRODUCTION

The human body made up of several organ systems that work together as one unit. The organ systems contain a network of specialized cells called neurons. A neuron or a nerve cell is an electrically exit able cell that transmit information between different parts by electrical, chemical signal and coordinate the actions of the body. The nervous system can be found in only some kinds of animals especially in multi-cellular animals. However, the structure of a nervous system that responds to activation from the world outside to all other nervous cells can be found in vertebrates. At the most basic level, the function of the nervous system is to send signals from one cell to other cells or from one part to other parts of the body. Neurons in the nervous system send signals to other cells

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as electrochemical waves traveling along thin fibers called axons, which cause chemicals called neurotransmitters to be released at junctions called synapses.

When a cell that receives a synaptic signal, it may be excited, inhibited, or otherwise modulated. Normally, the human brain has approximately billion of neurons and 10^{15}synapses. Each neuron is connected to form complicated network system. Nervous cells have efficiency to carry and receive messages or impulses as electrical or chemical signal quickly from one part to another part of the body. These impulses may carry information from the outside world to the nervous system which allows the body to quickly respond them. Abnormal electrical discharges from brain cells result in a recurrent seizure disorder such as epilepsy and migraine. So, the progress in understanding spatially structured seen in neural tissues such as how the activity patterns are generated is very important. An analysis of theoretical models for networks of nerve cells is necessary and the new types of medications to treat some neurological diseases may be suggested.

MATHEMATICAL MODEL

In this research, we investigate the case of a mutually inhibitory layer of cells coupled via inhibitory connections to a second layer of cells that provides excitatory feedback connections to the first layer. We derive results on the qualitative behavior of wave fronts conditional on parameters that represent the excitation threshold and time scale of the excitation process. The general neural field model, which is proposed by many works such as [1-5] takes the form

$$u_t(x,t)+u(x,t)=\alpha_1\int_{\Re}K_1(x-y)H(u(y,t)-\theta_1)dy-\alpha_2\int_{\Re}K_2(x-y)H(v(y,t)-\theta_2)dy \quad (1)$$

$$\tau v_t(x,t)+v(x,t)=\int_{\Re}K_3(x-y)H(u(y,t)-\theta_3)dy \quad . \quad (2)$$

where $u(x,t)$ represents the potential of a cell in the first layer located at x at time t, and $v(x,t)$ represents the potential of a cell in the second layer at location x at time t, then the neural network of interest is given, for $(x,t)\in\Re\times\Re^+$. The equations (1) and (2) are considered non-dimensional, with $\tau>0$ representing the ratio of time scales associated with u and v. We will make a further assumption that the spread of excitatory influence from layer 1 to layer 2 is very narrow. This allows us to idealize the excitatory connection function to $K_3(x)=k_3\delta(x)$, where $\delta(x)$ is the Dirac delta function. Thus, equation (2) becomes

$$\tau v_t(x,t)+v(x,t)=k_3H(u(x,t)-\theta_3) \quad . \quad (3)$$

A further reduction we do here is to let $\theta_1 = \theta_3 = \theta$. We want to look for solutions of the form $\big(u(x,t), v(x,t)\big) = \big(U(z), V(z)\big),\ z = x + \upsilon t$, for some wave speed $\upsilon > 0$.

Here, we assume that the following condition holds:

(A1) $K_j(\cdot)$, $j = 1,2$, are positive, even, smooth, single humped functions on $\Re$, with $\int_{\Re} K_j(x)dx = 1$, $\int_{\Re} |K_j'(x)|dx < \infty$, and exponentially decay for $|x|$ sufficiently large. Also, $K_1(x) < K_2(x)$ for $|x| < x_0$, and $K_1(x) > K_2(x)$ for $|x| > x_0$.

(A2) $U(0) = \theta$.

(A3) $U(z) > \theta$ if and only if $z > 0$.

By using the assumptions (A2) and (A3), we then have the non-trivial solutions, equations (1) and (3) thus become

$$\upsilon U' + U = \alpha_1 \int_{-\infty}^{z} K_1(x)dx - \alpha_2 \int_{\Re} K_2(z-y)H(V(y) - \theta_2)dy \tag{4}$$

and

$$\upsilon\tau V' + V = k_3 H(z). \tag{5}$$

The solution to equation (5) is

$$V(z) = k_3\left(1 - e^{-\frac{z}{\upsilon\tau}}\right)H(z) \tag{6}$$

with $V'(z) = (\frac{k_3}{\upsilon\tau})e^{\frac{-z}{\upsilon\tau}}H(z) > 0$.

Then, there exists a unique $z = z_1 > 0$ such that $V(z_1) = \theta_2$, so that $\theta_2 = k_3\left(1 - e^{-\frac{z_1}{\upsilon\tau}}\right)$ or $z_1 = \tau\upsilon \ln\left[\frac{k_3}{k_3 - \theta_2}\right]$ provided $k_3 > \theta_2$. Now, for $\upsilon > 0$, equation (4) thus becomes

$$\upsilon U' + U = \alpha_1 \int_{-\infty}^{z} K_1(x)dx - \alpha_2 \int_{-\infty}^{z-z_1} K_2(x)dx \tag{7}$$

and the solution to equation (7) becomes

$$U(z) = \alpha_1 \int_{-\infty}^{z} K_1(x)dx - \alpha_1 \int_{-\infty}^{z} e^{\frac{(x-z)}{\upsilon}} K_1(x)dx - \alpha_2 \int_{-\infty}^{z-z_1} K_2(x)dx + \alpha_2 \int_{-\infty}^{z} e^{\frac{(x-z)}{\upsilon}} K_2(x - z_1)dx. \tag{8}$$

Since z_1 depends linearly on τ, as $\tau \to 0^+$, $z_1 \to 0$. In this limit, equation (7) reduces to

$$\upsilon U' + U = \int_{-\infty}^{z} \left(\alpha_1 K_1(x) - \alpha_2 K_2(x)\right) dx = \alpha \int_{-\infty}^{z} K(x) dx \quad (9)$$

If $\alpha_1 K_1(x)$ has smaller amplitude and wider spread than $\alpha_2 K_2(x)$, then $\alpha K(x) = \alpha_1 K_1(x) - \alpha_2 K_2(x)$ in equation (9) is a lateral-excitation type kernel.

Analysis of wave shape with respect to $z_1(\tau)$

We shall show that the wave front is a monotone function on $(-\infty, 0)$ (Lemma 1). In addition, the wave shape is monotone on $(0, \infty)$ when τ is close to 0 (Lemma 2). On the other hand, for τ sufficiently large, the wave has a local maximum point in (x_0, z_1), that is, it is an increasing function on $(0, x_0)$. It then decreases to its positive asymptotic value as $z \to +\infty$ (Lemma 3).

Lemma 1 For $0 < 2\theta < \alpha_1 - \alpha_2$, $0 < \theta_2 < k_3$ and for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $\tau > \dfrac{\delta}{\alpha_2 \inf\limits_{(-\infty,z)} K_2'(x) \upsilon \ln\left[\dfrac{k_3}{k_3 - \theta_2}\right]} \equiv \tau_M^* > \tau_m^*$, $z < 0$, $U'(z) > 0$ on $z < 0$.

Proof. From equation (8), we have

$$U'(z) = \frac{1}{\upsilon} \int_{-\infty}^{z} e^{\left(\frac{x-z}{\upsilon}\right)} \left\{\alpha_1 K_1(x) - \alpha_2 K_2(x - z_1)\right\} dx \quad (10)$$

For $z \in (-\infty, 0)$, then, for $\varepsilon \in (0, -z)$, there is a $\delta > 0$ such that for $x \in (-\infty, z + \varepsilon)$, $\alpha_2 K_2(x) - \alpha_1 K_1(x) < \delta$.

Letting $\inf\limits_{(-\infty,z)} K_2'(x) = \Delta$ (which is positive), for $\tau > \dfrac{\delta}{\alpha_2 \inf\limits_{(-\infty,z)} K_2'(x) \upsilon \ln\left[\dfrac{k_3}{k_3 - \theta_2}\right]}$, this implies $z_1 > \dfrac{\delta}{\alpha_2 \Delta}$, and for some $\psi \in (x - z_1, x)$, we have

$$\begin{aligned} \alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) &= \alpha_1 K_1(x) - \alpha_2 \left\{K_2(x) - z_1 K_2'(\psi)\right\} \\ &> \alpha_1 K_1(x) - \alpha_2 K_2(x) + \delta \end{aligned}$$

Thus, $\displaystyle\int_{-\infty}^{z} e^{\left(\frac{x-z}{\upsilon}\right)} \left\{\alpha_1 K_1(x) - \alpha_2 K_2(x - z_1)\right\} dx > \int_{-\infty}^{z} e^{\left(\frac{x-z}{\upsilon}\right)} \left\{\alpha_1 K_1(x) - \alpha_2 K_2(x)\right\} dx + \delta \int_{-\infty}^{z} e^{\left(\frac{x-z}{\upsilon}\right)} dx > 0$, giving $U'(z) > 0$ for $x \in (-\infty, z + \varepsilon)$. Since this holds for all $\varepsilon \in (0, -z)$, then $U'(z) > 0$ for $x \in (-\infty, z)$. Thus, $U'(z) > 0$ on $z < 0$. □

Lemma 2 For $0 < 2\theta < \alpha_1 - \alpha_2$, θ_2 sufficiently small, and for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < \tau < \dfrac{-\delta}{\alpha_2 \inf\limits_{(0,z)} K_2'(x) \upsilon \ln\left[\dfrac{k_3}{k_3 - \theta_2}\right]}$, $z > 0$, $U'(z) > 0$ on $z > 0$.

Proof. From equation (8), we obtain

$$\upsilon U'(z) e^{\frac{z}{\upsilon}} = \left\{ \frac{\alpha_1}{2} - \alpha_2 \int_{-\infty}^{-z_1} K_2(x)dx - \theta \right\} + \int_0^z e^{\frac{x}{\upsilon}} \{\alpha_1 K_1(x) - \alpha_2 K_2(x - z_1)\} dx$$

$$> \int_0^z e^{\frac{x}{\upsilon}} \{\alpha_1 K_1(x) - \alpha_2 K_2(x - z_1)\} dx .$$

For $z \in (0, \infty)$, then for any $\varepsilon \in (0, z)$, there is a $\delta > 0$ such that for $x \in (0, z - \varepsilon)$, $\alpha_1 K_1(x) - \alpha_2 K_2(x) > \delta$.

Letting $\inf\limits_{(0,z)} K_2'(x) = \Delta$ (which is negative), for $\tau < \dfrac{-\delta}{\alpha_2 \inf\limits_{(0,z)} K_2'(x) \upsilon \ln\left[\dfrac{k_3}{k_3 - \theta_2}\right]}$,

this implies $z_1 < \dfrac{-\delta}{\alpha_2 \Delta}$, and for some $\psi \in (x - z_1, x)$, we have

$$\alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) = \alpha_1 K_1(x) - \alpha_2 \{K_2(x) - z_1 K_2'(\psi)\}$$

$$> \alpha_1 K_1(x) - \alpha_2 K_2(x) - \delta .$$

Thus,

$$\int_0^z e^{\frac{x}{\upsilon}} \{\alpha_1 K_1(x) - \alpha_2 K_2(x - z_1)\} dx > \int_0^z e^{\frac{x}{\upsilon}} \{\alpha_1 K_1(x) - \alpha_2 K_2(x)\} dx - \delta \int_0^z e^{\frac{x}{\upsilon}} dx > 0 , ,$$

giving $\upsilon U'(z) e^{\frac{z}{\upsilon}} > 0$ for $x \in (0, z - \varepsilon)$. Since this holds for all $\varepsilon \in (0, z)$, then $\upsilon U'(z) e^{\frac{z}{\upsilon}} > 0$ for $x \in (0, z)$. Thus, $U'(z) > 0$ on $z > 0$. □

Lemma 3 For $0 < 2\theta < \alpha_1 - \alpha_2$, $0 < \theta_2 < k_3$, assume, besides (A1), that $K_1(x)$ and $K_2(x)$ satisfy the followings.

There exist $m_1, m_2, \rho_1, \rho_2 > 0$, such that $K_1(x) < m_1 e^{-\rho_1 x}$, $K_2(x) < m_2 e^{-\rho_2 x}$ for $x > 0$.

$\upsilon \neq \dfrac{1}{\rho_1}$ and $\upsilon \neq \dfrac{1}{\rho_2}$.

$\inf\limits_{[0,\infty)} K_2'(x) > -\infty$.

Then, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $\tau < \dfrac{-\delta}{\alpha_2 \inf\limits_{(0,x_0)} K_2'(x)\upsilon \ln\left[\dfrac{k_3}{k_3-\theta_2}\right]} \equiv \tau_M^{**}$, there is a unique $z = z_2 \in (x_0, z_1)$ which is a local maximum point for $U(z)$ in that $U'(z) > 0$ on $[0, z_2)$, and $U'(z) < 0$ on (z_2, ∞).

Proof. For $z \in (0, x_0)$, from equation (8), we obtain

$$\upsilon U'(z)e^{\frac{z}{\upsilon}} = \left\{\frac{\alpha_1}{2} - \alpha_2 \int_{-\infty}^{-z_1} K_2(x)dx - \theta\right\} + \int_0^z e^{\frac{x}{\upsilon}}\{\alpha_1 K_1(x) - \alpha_2 K_2(x-z_1)\}dx$$
$$> \int_0^z e^{\frac{x}{\upsilon}}\{\alpha_1 K_1(x) - \alpha_2 K_2(x-z_1)\}dx.$$

Then, for $\varepsilon \in (0, x_0)$, there is a $\delta > 0$ such that for $x \in (0, x_0 - \varepsilon)$, $\alpha_1 K_1(x) - \alpha_2 K_2(x) > \delta$.

Letting $\inf\limits_{(0,x_0)} K_2'(x) = \Delta < 0$, for $\tau < \dfrac{-\delta}{\alpha_2 \inf\limits_{(0,x_0)} K_2'(x)\upsilon \ln\left[\dfrac{k_3}{k_3-\theta_2}\right]}$, this implies $z_1 < \dfrac{-\delta}{\alpha_2 \Delta}$, and for some $\psi \in (x - z_1, x)$, we have

$$\alpha_1 K_1(x) - \alpha_2 K_2(x-z_1) = \alpha_1 K_1(x) - \alpha_2\{K_2(x) - z_1 K_2'(\psi)\}$$
$$> \alpha_1 K_1(x) - \alpha_2 K_2(x) - \delta.$$

Thus,

$$\int_0^z e^{\frac{x}{\upsilon}}\{\alpha_1 K_1(x) - \alpha_2 K_2(x-z_1)\}dx > \int_0^z e^{\frac{x}{\upsilon}}\{\alpha_1 K_1(x) - \alpha_2 K_2(x)\}dx - \delta\int_0^z e^{\frac{x}{\upsilon}}dx > 0,$$

giving $U'(z) > 0$ on $z \in (0, x_0 - \varepsilon)$. Since this holds for all $\varepsilon \in (0, x_0)$, then $U'(z) > 0$ on $z \in (0, x_0)$. Thus, $U(z)$ is monotone increasing on $(0, x_0)$.

If τ is sufficiently large, then $z_1 = \tau\upsilon \ln\left[\dfrac{k_3}{k_3-\theta_2}\right] >> 1$ and $z_1(\tau) > x_0$. When $z \in (z_1, \infty)$,

$$\upsilon U'(z)e^{\frac{z}{\upsilon}} = \alpha_1 \int_{-\infty}^{z} e^{\frac{x}{\upsilon}} K_1(x)dx + \alpha_2 e^{\frac{z_1}{\upsilon}} \int_{-\infty}^{z-z_1} e^{\frac{x}{\upsilon}} K_2(x)dx$$
$$< \alpha_1 \int_{-\infty}^{z} e^{\frac{x}{\upsilon}}\{m_1 e^{-\rho_1 x}\}dx + \alpha_2 e^{\frac{z_1}{\upsilon}} \int_{-\infty}^{z-z_1} e^{\frac{x}{\upsilon}}\{m_2 e^{-\rho_2 x}\}dx$$
$$= \frac{\alpha_1 m_1 \upsilon}{1-\rho_1\upsilon}\left\{e^{\left(\frac{1}{\upsilon}-\rho_1\right)z} - 1\right\} + \frac{\alpha_2 m_2 e^{\frac{z_1}{\upsilon}}\upsilon}{1-\rho_2\upsilon}\left\{e^{\left(\frac{1}{\upsilon}-\rho_2\right)(z-z_1)} - 1\right\}$$

That is, $\upsilon U'(z) < \dfrac{\alpha_1 m_1 \upsilon}{1-\rho_1\upsilon}\left\{e^{-\rho_1 z} - e^{-\frac{z}{\upsilon}}\right\} + \dfrac{\alpha_2 m_2 e^{\frac{z_1}{\upsilon}}\upsilon}{1-\rho_2\upsilon}\left\{e^{-\rho_2(z-z_1)-\frac{z_1}{\upsilon}} - e^{-\frac{z}{\upsilon}}\right\}$. If $z > z_1$ and $z \to +\infty$, then $e^{-\rho_2(z-z_1)-\frac{z_1}{\upsilon}} - e^{-\frac{z}{\upsilon}} \to 0$. For $z \in (z_1,\infty)$. Hence $U'(z) < 0$ on (z_1,∞), given conditions 1)-3). So, there exists a $z = z_2$, $z_2 \in (x_0, z_1)$, such that $U'(z_2) = 0$ and $U(z_2)$ is a local maximum. Moreover, we can prove that z_2 is unique but it is obviously so we will omit.

Example Consider $K_1(x) = a_1 e^{-b_1|x|}$ and $K_2(x) = a_2 e^{-b_2|x|}$ where $0 < a_1 < a_2$, $0 < b_1 < b_2$, $a_1 = \dfrac{b_1}{2}$ and $a_2 = \dfrac{b_2}{2}$. Now, from equation (8), for a numerical example, we let $a_1 = 1, b_1 = 2, a_2 = 2, b_2 = 4$, $\alpha_1 = 3$ and $\alpha_2 = 1$. Then, $K_1(x) = e^{-2|x|}$ and $K_2(x) = 2e^{-4|x|}$, whose graphs are shown in Figure 1.

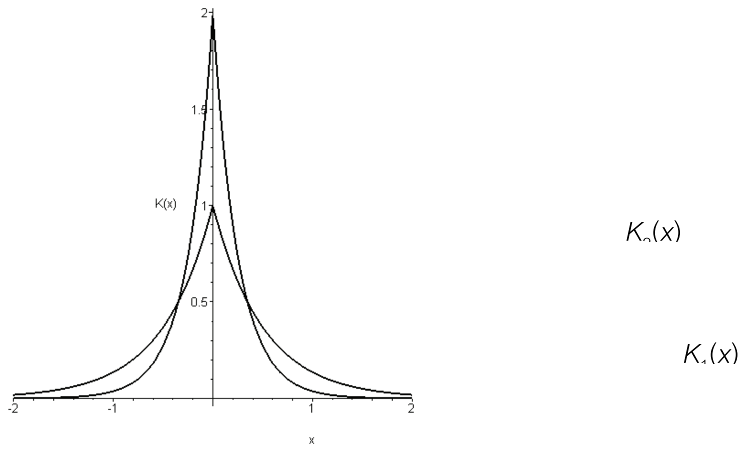

Figure 1 The graphs of $K_1(x) = e^{-2|x|}$ and $K_2(x) = 2e^{-4|x|}$.

Now, $e^{-2x_0} - 2e^{-4x_0} = 0$, yields $x_0 = \dfrac{\ln 2}{2} \approx 0.347 > 0$. In order to satisfy the conditions in Lemma 1 and Lemma 2, we let $\tau = 0.01$, $\theta_2 = 0.1$, $\theta = 0.245$ and $k_3 = 0.9$. Thus,

$$z_1 = \tau\upsilon \ln\left[\frac{k_3}{k_3-\theta_2}\right] = 0.01\upsilon\ln(1.125).$$

From equation, $\theta \equiv A(\upsilon) = \dfrac{\alpha_1}{2} - \alpha_1\int_{-\infty}^{0} e^{\frac{x}{\upsilon}} K_1(x)dx - \alpha_2\int_{-\infty}^{-z_1} K_2(x)dx + \alpha_2 e^{\frac{z_1}{\upsilon}}\int_{-\infty}^{-z_1} e^{\frac{x}{\upsilon}} K_2(x)dx$ yielding $\upsilon = 2$. The function $U(z)$ is monotone increasing function as shown in Figure 2.

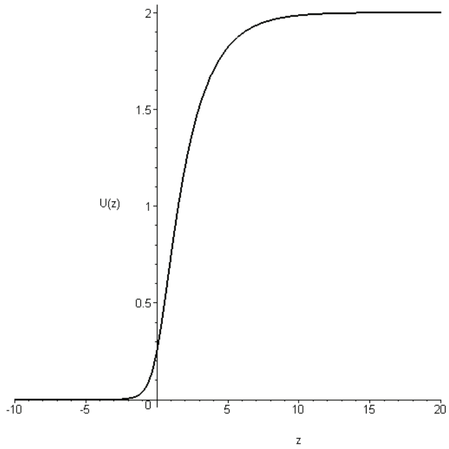

Figure 2. The graph of $U(z)$ with $\theta_2 = 0.1$, $\tau = 0.01$, $x_0 = 0.347$, $z_1 = 0.00236$ and $\upsilon = 2$.

In order to satisfy the conditions in Lemma 1 and Lemma 3, we let $\tau = 100$, $k_3 = 0.9$, $\rho_1 = \rho_2 = 1$, $1 > m_1 > e^{-x}$ and $1 > m_2 > 2e^{-3x}$ for $x > 0$. Then, $K_1(x) = e^{-2x} < m_1 e^{-\rho_1 x}$ and $K_2(x) = 2e^{-4x} < m_2 e^{-\rho_2 x}$ on $x \in (x_0, \infty)$. Also, $\upsilon \neq \frac{1}{\rho_1}, \frac{1}{\rho_2}$, that is, $\upsilon \neq 1$. Thus, if we choose $\theta_2 = 0.1$ and $\theta = 0.3$, then $U(z)$ is monotone for $z < 0$ but non-monotone for $z > 0$ as shown in Figure 3.

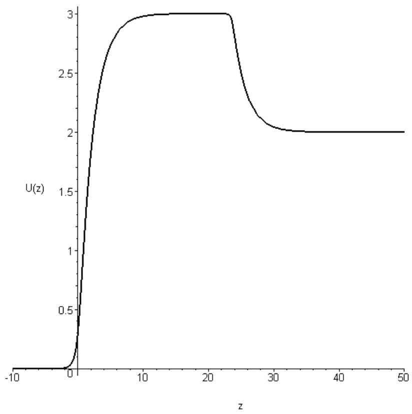

Figure 3 The graph of $U(z)$ with $\theta_2 = 0.1$, $\tau = 100$, $x_0 = 0.347$, $z_1 = 23.5566$ and $\upsilon = 2$.

CONCLUSION AND DISCUSSION

In this study we have analyzed the shape of traveling wave front solutions for a double continuous layer of nerve cells. We found that the waves are stable for all cases.

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