

# Some geometric properties of Musielak-Orlicz direct sums space.

Winate sanhan

Department of Mathematics, Faculty of Liberal Arts and science, Kasetsart University, Kamphaeng Saen Campus, Nakhonpathom 73140, Thailand

*Email : winate\_s@yahoo.com*

## Abstract

In this research we give some characterizations of H-property and Schur's property on the Musielak-Orlicz direct sums space  $X_1 \oplus_M \dots \oplus_M X_n$ . We prove that it has property (H) if and only if each  $X_i$  has property (H), and it has Schur's property if and only if each  $X_i$  has Schur's property.

## Introduction

In 1998, Megginson [5] introduced the new spaces by interested about a direct sums of Banach spaces, namely  $X_1 \oplus_2 \dots \oplus_2 X_n$  defined as follows ; Let  $(X_1, \|\cdot\|_1), \dots, (X_n, \|\cdot\|_n)$  be the normed spaces . The *direct sums or direct products* of  $X_1, \dots, X_n$ , defined by  $X_1 \oplus_2 \dots \oplus_2 X_n = \{x = (x_1, \dots, x_n) / \sum_{j=1}^n \|x_j\|_j^2 < \infty\}$  equipped with

$$\|(x_1, \dots, x_n)\|_2 = \left( \sum_{j=1}^n \|x_j\|_j^2 \right)^{\frac{1}{2}}. \quad (1)$$

He show that the spaces  $X_1 \oplus_2 \dots \oplus_2 X_n$  equipped with a norm  $\|\cdot\|_2$  is Banach if and only if each  $X_i$  is Banach. Moreover, he give characterization of rotundity of  $X_1 \oplus_2 \dots \oplus_2 X_n$  ; such as  $X_1 \oplus_2 \dots \oplus_2 X_n$  is rotund if and only if each  $X_i$  is rotund.

After that, the generalized of direct sums spaces was defined in many way. For example in 2003, K. S. Saito and M. Kato defined the direct sums of Banach spaces by using continuous convex function and study some geometric property on its. This extends the direct sum  $X_1 \oplus_p \dots \oplus_p X_n = \{x = (x_1, \dots, x_n) / \sum_{j=1}^n \|x_j\|_j^p < \infty\}$

$x_j \|_j^p < \infty$ , ( $1 \leq p \leq \infty$ ) equipped with

$$\| (x_1, \dots, x_n) \|_p = \left( \sum_{j=1}^n \| x_j \|_j^p \right)^{\frac{1}{p}}$$

and convexity properties of the direct sums are characterized.

In 2005, R. Poonchaay [6] defined the Orlicz direct sums of Banach spaces  $X_1 \oplus_M \dots \oplus_M X_n$  where  $M$  is Orlicz function and study this space under the Luxemburg norm and give characterization of rotundity and also give characterization of property (H) and Schur's property.

In this study we will generalized the direct sums spaces in the sense of the Musielak-Orlicz function.

Let  $M = (M_i)_{i=1}^n$  be a Musielak-Orlicz function and  $(X_1, \|\cdot\|_1), \dots, (X_n, \|\cdot\|_n)$  be a Banach spaces. The Musielak-Orlicz direct sums of  $(X_1, \|\cdot\|_1), \dots, (X_n, \|\cdot\|_n)$ , defined by  $X_1 \oplus_M \dots \oplus_M X_n = \{x = (x_1, \dots, x_n) / \sum_{i=1}^n M_i(\| x_i \|_i) < \infty\}$  equipped with

$$\|(x_1, \dots, x_n)\|_M = \inf \{ \lambda > 0 : \varrho_M \left( \frac{(x_1, \dots, x_n)}{\lambda} \right) \leq 1 \}$$

where

$$\varrho_M(x) = \sum_{i=1}^n M_i(\| x_i \|_i).$$

**Definition 1.** For a real vector space  $X$ , a function  $\varrho : X \rightarrow [0, \infty]$  is called a *modular* if it satisfies the following conditions :

- (i)  $\varrho(x) = 0$  if and only if  $x = 0$ ;
- (ii)  $\varrho(\alpha x) = \varrho(x)$  for all scalar  $\alpha$  with  $|\alpha| = 1$ ;
- (iii)  $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$ , for all  $x, y \in X$  and all  $\alpha, \beta \geq 0$  with  $\alpha + \beta = 1$ .

The modular  $\varrho$  is called convex if

- (iv)  $\varrho(\alpha x + \beta y) \leq \alpha \varrho(x) + \beta \varrho(y)$ , for all  $x, y \in X$  and all  $\alpha, \beta \geq 0$  with  $\alpha + \beta = 1$ .

If  $\varrho$  is modular in  $X$ , we define

$$X_\rho = \{x \in X : \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0\},$$

and

$$X_\rho^* = \{x \in X : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

It clear that  $X_\rho \subseteq X_\rho^*$ . If  $\rho$  is a convex modular, for  $x \in X_\rho$  we define

$$\|x\| = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\} \tag{2}$$

### Orlicz direct sums spaces

Recall that the *vector space sum* of the vector spaces  $X_1, \dots, X_n$  in a nonempty finite ordered list is the vector space whose underlying set Cartesian product  $X_1 \times \dots \times X_n$  and which as the vector space operations given by the formulas

$$\begin{aligned} (x_1, \dots, x_n) + (y_1, \dots, y_n) &= (x_1 + y_1, \dots, x_n + y_n) \\ \alpha \cdot (x_1, \dots, x_n) &= (\alpha x_1, \dots, \alpha x_n). \end{aligned}$$

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$$\| (x_1, \dots, x_n) \|_2 = \left( \sum_{j=1}^n \|x_j\|_j^2 \right)^{\frac{1}{2}}. \tag{3}$$

He show that the spaces  $X_1 \oplus_2 \dots \oplus_2 X_n$  equipped with a norm  $\|\cdot\|_2$  is Banach if and only if each  $X_i$  is Banach. Moreover, he give characterization of rotundity of  $X_1 \oplus_2 \dots \oplus_2 X_n$  ; such as  $X_1 \oplus_2 \dots \oplus_2 X_n$  is rotund if and only if each  $X_i$  is rotund.

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on its. This extends the direct sum  $X_1 \oplus_p \dots \oplus_p X_n = \{x = (x_1, \dots, x_n) / \sum_{j=1}^n \|x_j\|_j^p < \infty\}$ , ( $1 \leq p \leq \infty$ ) equipped with

$$\|(x_1, \dots, x_n)\|_p = \left(\sum_{j=1}^n \|x_j\|_j^p\right)^{\frac{1}{p}}$$

and convexity properties of the direct sums are characterized.

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In this study we will generalized the direct sums spaces in the sense of the Musielak-Orlicz function.

Let  $M = (M_i)_{i=1}^n$  be a Musielak-Orlicz function and  $(X_1, \|\cdot\|_1), \dots, (X_n, \|\cdot\|_n)$  be a Banach spaces. The Musielak-Orlicz direct sums of  $(X_1, \|\cdot\|_1), \dots, (X_n, \|\cdot\|_n)$ , defined by  $X_1 \oplus_M \dots \oplus_M X_n = \{x = (x_1, \dots, x_n) / \sum_{i=1}^n M_i(\|x_i\|_i) < \infty\}$  equipped with

$$\|(x_1, \dots, x_n)\|_M = \inf\{\lambda > 0 : \varrho_M\left(\frac{(x_1, \dots, x_n)}{\lambda}\right) \leq 1\}$$

where

$$\varrho_M(x) = \sum_{i=1}^n M_i(\|x_i\|_i).$$

## The relationship between Modular and Norm.

**Proposition 2.** *If  $X_1, \dots, X_n$  are normed spaces and each  $M_i : \mathbb{R} \rightarrow \mathbb{R}$  is an Orlicz function, then  $\varrho_M$  is a convex modular on  $X_1 \oplus_M \dots \oplus_M X_n$ .*

**Proposition 3.** *Let  $x \in X_1 \oplus_M \dots \oplus_M X_n$ . Then*

- (1) *For  $\alpha \geq 1$  implies  $\varrho_M(x) \geq \alpha \varrho_M\left(\frac{x}{\alpha}\right)$ .*
- (2) *For  $0 < \alpha < 1$  implies  $\varrho_M(x) \leq \alpha \varrho_M\left(\frac{x}{\alpha}\right)$ .*
- (3) *For  $\alpha \geq 1$  implies  $\varrho_M(\alpha x) \geq \alpha \varrho_M(x)$ .*
- (4) *For  $0 < \alpha < 1$  implies  $\varrho_M(\alpha x) \leq \alpha \varrho_M(x)$ .*

**Proposition 4.** let  $x \in X_1 \oplus_M \dots \oplus_M X_n$

- (1)  $\|x\|_M \leq 1$  implies  $\varrho_M(x) \leq \|x\|_M$
- (2)  $\|x\|_M > 1$  implies  $\varrho_M(x) \geq \|x\|_M$
- (3)  $\|x\|_M = 1$  if and only if  $\varrho_M(x) = 1$ .

**Theorem 5.** Let  $(x_k)$  be a sequence in the space  $X_1 \oplus_M \dots \oplus_M X_n$ . Then

$$\|x_k\|_M \rightarrow 0 \text{ if and only if } \varrho_M(x_k) \rightarrow 0.$$

**Lemma 6.** Let  $x \in X_1 \oplus_M \dots \oplus_M X_n$  and  $(x_k)$  be a sequence in the space  $X_1 \oplus_M \dots \oplus_M X_n$ . Then

- (1)  $\|x_k\|_M \rightarrow \infty$  if and only if  $\varrho_M(x_k) \rightarrow \infty$ .
- (2) for any  $\epsilon > 0$ , there exists  $\beta > 0$  such that  $\|x\|_M \geq \epsilon$  implies  $\varrho_M(x) \geq \beta$ .
- (3) for any  $\epsilon > 0$ , there exists  $\beta \in (0, 1)$  such that  $\varrho_M(x) \leq 1 - \epsilon$  implies  $\|x\|_M \leq 1 - \beta$ .

### Property (H) and Schur 's Property.

**Theorem 7.**  $X_1 \oplus_M \dots \oplus_M X_n$  has the property(H) if and only if each  $X_i$  has the property(H).

**Proof.** *Necessity.* If  $X_1 \oplus_M \dots \oplus_M X_n$  has the property(H), then  $X_i$  is a subspace of  $X_1 \oplus_M \dots \oplus_M X_n$ . Since it is isometrically isomorphic to  $X_i'$ . So  $X_i$  has the property(H).

*Sufficiency.* Assume that each  $X_i$  has the property(H). Letting  $(x_k)$  be the sequence in the  $S(X_1 \oplus_M \dots \oplus_M X_n)$  and  $x \in S(X_1 \oplus_M \dots \oplus_M X_n)$  such that  $x_k \xrightarrow{w} x$ .

(1) We will show for each  $i = 1, \dots, n, x_i^k \xrightarrow{w} x_i$ . Put  $T_i$  be element in  $X_i'$ . We define a function  $f : X_1 \oplus_M \dots \oplus_M X_n \rightarrow \mathbb{R}$  by  $f(x) = T_i(x_i)$ . We will show that  $f \in (X_1 \oplus_M \dots \oplus_M X_n)'$ .

Since for any  $\alpha, \beta \in \mathbb{R}$  and  $x, y \in X_1 \oplus_M \dots \oplus_M X_n$ , We have

$$\begin{aligned} f(\alpha x + \beta y) &= f((\alpha x_i + \beta y_i, \dots, \alpha x_n + \beta y_n)) \\ &= T_i(\alpha x_i + \beta y_i) \\ &= \alpha T_i(x_i) + \beta T_i(y_i) \\ &= \alpha f(x) + \beta f(y). \end{aligned}$$

That is  $f$  is linear.

Giving  $x \in X_1 \oplus_M \dots \oplus_M X_n$  and  $\varepsilon > 0$ . Choose  $\delta = \frac{\varepsilon}{M_i^{-1}(1)\|T_i\|}$

let  $x' \in X_1 \oplus_M \dots \oplus_M X_n$  such that  $\|x - x'\|_M < \delta$ . Then  $\|\frac{x - x'}{\delta}\|_M < 1$ .

By proposition 16(1),  $\varrho_M(\frac{x - x'}{\delta}) \leq 1$ . Then  $\varrho_M(\frac{(x_i - x'_i, \dots, x_n - x'_n)}{\delta}) \leq 1$ .

So  $\sum_{i=1}^n M_i(\frac{\|x_i - x'_i\|_i}{\delta}) \leq 1$ .

Hence  $M_i(\frac{\|x_i - x'_i\|_i}{\delta}) \leq 1$  for all  $i=1, \dots, n$ . Thus  $\|x_i - x'_i\|_i \leq \delta M_i^{-1}(1)$ . Hence we consider

$$\begin{aligned} |f(x) - f(x')| &= |T_i(x_i) - T_i(x'_i)| \\ &= |T_i(x_i - x'_i)| \\ &\leq \|T_i\| \|x_i - x'_i\|_i \\ &\leq \|T_i\| \delta M_i^{-1}(1) \\ &= \|T_i\| \cdot \frac{\varepsilon}{M_i^{-1}(1)\|T_i\|} \cdot M_i^{-1}(1) \\ &= \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Thus  $f$  is continuous. Therefore  $f \in (X_1 \oplus_M \dots \oplus_M X_n)'$ .

Since  $x_k \xrightarrow{w} x$  and  $f \in (X_1 \oplus_M \dots \oplus_M X_n)'$ ,

$f(x_k) \rightarrow f(x)$ . By a definition of  $f$ , we have  $T_i(x_i^k) \rightarrow T_i(x_i)$ .

Therefore  $(x_i^k)$  converges weakly to  $x_i$  for all  $i = 1, \dots, n$ .

(2) We will show for each  $i = 1, \dots, n$ ,  $\|x_i^k\|_i \rightarrow \|x_i\|_i$

Since  $x_i^k \xrightarrow{w} x_i$  for all  $i = 1, \dots, n$ , it implies

$$\|x_i\|_i \leq \liminf_{k \rightarrow \infty} \|x_i^k\|_i \text{ for all } i = 1, \dots, n.$$

Next we claim that  $\limsup_{k \rightarrow \infty} \|x_i^k\|_i \leq \|x_i\|_i$  for all  $i = 1, \dots, n$ .

Suppose there is  $i_0$  such that  $\limsup_{k \rightarrow \infty} \|x_{i_0}^k\|_{i_0} > \|x_{i_0}\|_{i_0}$ . Then there exists  $\varepsilon > 0$  and  $N_1 \in \mathbb{N}$  such that  $\|x_{i_0}^k\|_{i_0} \geq \|x_{i_0}\|_{i_0} + \varepsilon$  for all  $k \geq N_1$ .

Since  $\|x_i\|_i \leq \liminf_{k \rightarrow \infty} \|x_i^k\|_i$ , then there  $N_2 \in \mathbb{N}$  such that

$\|x_i\|_i \leq \|x_i^k\|_i$  for all  $k \geq N_2$  and for all  $i = 1, \dots, n$ .

Hence, for  $k \geq N = \max\{N_1, N_2\}$

$$\begin{aligned} 1 = \varrho(x_k) &= \sum_{i=1}^n M_i(\|x_i^k\|_i) \\ &\geq \sum_{i \neq i_0}^n M_i(\|x_i^k\|_i) + M_{i_0}(\|x_{i_0}\|_{i_0} + \varepsilon) \\ &\geq \sum_{i \neq i_0}^n M_i(\|x_i\|_i) + M_{i_0}(\|x_{i_0}\|_{i_0} + \varepsilon) \\ &> \sum_{i \neq i_0}^n M_i(\|x_i\|_i) + M_{i_0}(\|x_{i_0}\|_{i_0}) \\ &= \varrho(x) = 1 \end{aligned}$$

such that it contradict with fact.

Hence  $\limsup_{k \rightarrow \infty} \|x_i^k\|_i \leq \|x_i\|_i$  for all  $i = 1, \dots, n$ .

Thus  $\limsup_{k \rightarrow \infty} \|x_i^k\|_i = \liminf_{k \rightarrow \infty} \|x_i^k\|_i = \|x_i\|_i$  for all  $i = 1, \dots, n$ .

Therefore  $\|x_i^k\|_i \rightarrow \|x_i\|_i$  for all  $i = 1, \dots, n$ .

Since for all  $i = 1, \dots, n$ ,  $x_i^k \xrightarrow{w} x_i$  and  $\|x_i^k\|_i \rightarrow \|x_i\|_i$ , by the property(H) of  $X_i$ ,  $x_i^k \rightarrow x_i$ . Thus  $x_k \rightarrow x$ . Therefore,  $X_1 \oplus_M \dots \oplus_M X_n$  has the property(H).

□

**Theorem 8.**  $X_1 \oplus_M \dots \oplus_M X_n$  has Schur's property if and only if each  $X_i$  has Schur's property.

**Proof.** *Necessity.* If  $X_1 \oplus_M \dots \oplus_M X_n$  has Schur's property, then each  $X_i$  has the Schur's property since it is isometrically isomorphic to a subspace of  $X_1 \oplus_M \dots \oplus_M X_n$ , it has Schur's property, then so does every its subspace.

*Sufficiency.* Assume that each  $X_i$  has the Schur's property.

Let  $(x_k)$  be the sequence in the space  $X_1 \oplus_M \dots \oplus_M X_n$  and

$x \in X_1 \oplus_M \dots \oplus_M X_n$  where  $x_k \xrightarrow{w} x$ . In proof of Theorem 11 we have  $x_i^k \xrightarrow{w} x_i$

for all  $i = 1, \dots, n$ . By using has *Schur's* property of  $X_i$ . Thus  $x_i^k \rightarrow x_i$  for all  $i = 1, \dots, n$ . Therefore  $x_k \rightarrow x$ .  $\square$

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