Nonlinear Impulsive Periodic Systems With Parameter Perturbations

Sujitra Sanhan

Abstract : We study the existence of periodic solutions for nonlinear impulsive periodic system with parameter perturbation on infinite dimension space, in these cases where the differential operator involved is the closed densely defined linear unbounded operator on Banach space.

Keywords : Impulsive differential equation, periodic solution, Banach space.

1 Introduction

The impulsive differential equations appear to a natural framework for mathematical modelings of several real world phenomena. For instance, systems with impulse effects have applications in physics, in biotechnology, in population dynamics, in optimal control and so on. For an introduction to the theory of impulsive systems, we refer the reader to see in [6]-[11]. In the framework of impulsive differential equations, some existence result of periodic solutions for impulsive periodic system with parameter perturbation on finite dimensional space has been studied by many authors in [2] and [10].

However, the investigation of existence of periodic solutions for nonlinear impulsive periodic systems with parameter perturbation on infinite dimensional space have not been study. We apply the semigroup theory (see [1] and [4]) and fixed point theorems (see [3]) for impulsive systems, we establish conditions for ensuring that the system has a unique periodic solution.

2 nonlinear Impulsive Periodic System and Impulsive Evolution Operator

Let $\mathcal{L}(X)$ be the space of bounded linear operators in the Banach space X.Define $PC([0,\infty),X) \equiv \{\, x : [0,\infty) \to X|\ \ x$ is continuous at $t \in [0,T_0],\ t \neq \tau_k,\ x$ is continuous from left and the right limit $x(\tau_k^+)$ exists at $t = \tau_k,\ \forall k \in \mathbb{N}\}$,$PC^1([0,\infty),X) \equiv \{x \in PC([0,\infty),X)|\ \dot{x} \in PC([0,\infty),X)\}$ and $PC_{T_0}([0,\infty),X) \equiv \{x \in PC([0,\infty),X)|\ x(t) = x(t+T_0),\ \forall t \geq 0\}$. where $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_{\sigma-1} < \tau_\sigma = T_0 < \infty$, which is a Banach space with the norm

$$\|x\|_{PC} = \sup_{t\in[0,T_0]} \|x(t)\|_X.$$

Consider the following nonlinear impulsive periodic systems

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t,x(t)), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t), & t = \tau_k, \end{cases} \tag{2.1}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$. Futhermore we suppose that $\{A(t), t \in [0,T_0]\}$ is a family of closed densely defined linear unbounded operator on X, satisfying the following assumptions (A1),(A2):

Assumption (A1) ;

(A1.1) $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots,\ \tau_k \to \infty$ as $k \to \infty$ and there exists a positive integer σ such that $\tau_{k+\sigma} = \tau_k + T_0$ for all $k \in \mathbb{N}$.

(A1.2) $B_k \in \mathcal{L}(X)$ such that $B_{k+\sigma} = B_k$ for all $k \in \mathbb{N}$ and there exists constant $h_k(\rho) > 0$ such that

$$\|B_k(x) - B_k(y)\|_X \leq h_k(\rho)\|x-y\|_X,$$

for all $k \in \mathbb{N}$ and all $x, y \in X$ such that $\|x\|_X,\ \|y\|_X \leq \rho$.

(A1.3) $f : [0,\infty) \times X \to X$ is an operator such that $f(t+T_0,x) = f(t,x)$ and $t \mapsto f(t,x)$ is strongly measurable. For every $\rho > 0$, there exist constants $K_1(\rho),\ K_2(\rho) > 0$ such that

$$\|f(t,x)\|_X \leq K_1(\rho) \qquad \text{and} \qquad \|f(t,x) - f(t,y)\|_X \leq K_2(\rho)\|x-y\|_X,$$

for all $t \geq 0$ and all $x, y \in X$ such that $\|x\|_X,\ \|y\|_X \leq \rho$.

Assumption (A2) ;

(A2.1) The domain $\mathcal{D}(A(t)) = \mathcal{D}$ is independent of t and dense in X for $t \in [0,T_0]$.

(A2.2) For $t \geq 0$ the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re\lambda \leq 0$, and there is a constant M of λ and t such that

$$\|R(\lambda, A(t))\|_{\mathcal{L}(X)} \leq M(1+|\lambda|)^{-1} \qquad \text{for all} \qquad Re\lambda \leq 0.$$

(A2.3) There exists constants $L > 0$ and $0 < \alpha \leq 1$ such that

$$\|(A(t) - A(s))A^{-1}(\tau)\|_{\mathcal{L}(X)} \leq L|t-s|^{\alpha} \qquad \text{for} \qquad t, s, \tau \in [0, T_0].$$

Lemma 2.1. (see [4], page 159).*Let assumption (A2) hold. The Cauchy problem*

$$\dot{x}(t) + A(t)x(t) = 0, \quad t \in (0, T_0] \qquad \textit{with} \qquad x(0) = x_0 \tag{2.2}$$

has a unique evolution system $\{U(t,s)|0 \leq s \leq t \leq T_0\}$ *in* X *satisfying the following properties :*

1. $U(t,s) \in \mathcal{L}(X)$, *for* $0 \leq s \leq t \leq T_0$ *;*
2. $U(t,r)U(r,s) = U(t,s)$, *for* $0 \leq s \leq r \leq t \leq T_0$ *;*
3. $U(.,.)x \in C(\triangle, X)$, *for* $x \in X, \triangle \equiv \{(t,s) \in [0,T_0] \times [0,T_0] | 0 \leq s \leq t \leq T_0\}$
4. *for* $0 \leq s \leq t \leq T_0, U(t,s) : X \to \mathcal{D}$ *and* $t \to U(t,s)$ *is strongly differentiable in* X*. The derivative* $(\frac{\partial}{\partial t})U(t,s) \in \mathcal{L}(X)$ *and it is strongly continuous on* $0 \leq s < t \leq T_0$ *; moreover*

$$\frac{\partial}{\partial t}U(t,s) = -A(t)U(t,s) \qquad \textit{for} 0 \leq s < t \leq T_0,$$

$$\|\frac{\partial}{\partial t}U(t,s)\|_{\mathcal{L}(X)} = \|A(t)U(t,s)\|_{\mathcal{L}(X)} \leq \frac{c}{t-s}$$

$$\|A(t)U(t,s)A(s)^{-1}\|_{\mathcal{L}(X)} \leq c \qquad \textit{for} \quad 0 \leq s \leq t \leq T_0;$$

5. *for every* $v \in \mathcal{D}$ *and* $t \in (0, T_0], U(t,s)v$ *is differentiable with respect to* s *on* $0 \leq s \leq t \leq T_0$

$$\frac{\partial}{\partial s}U(t,s)v = U(t,s)A(s)v.$$

And , for each $x_0 \in X$*, the Cauchy problem (2.2) has a unique classical solution* $x \in C^1([0,T_0], X)$ *given by*

$$x(t) = U(t,0)x_0, \qquad t \in [0, T_0].$$

In addition to Assumption (A2) , we introduce the following assumption.
Assumption (A3) ;
(A3.1) There exists $T_0 > 0$ such that $A(t+T_0) = A(t)$ for all$t \in [0, T_0]$.
(A3.2) For all $t \geq 0$, the resolvent $R(\lambda, A(t))$ is compact.

Lemma 2.2. (see [5], page 105). *Let assumption (A2) and (A3) hold. Then evolution system and* $\{U(t,s)|0 \leq s \leq t \leq T_0\}$ *in* X *also satisfying the following two properties :*

1. $U(t+T_0, s+T_0) = U(t,s)$, *for* $0 \leq s \leq t \leq T_0$ *;*
2. $U(t,s)$ *is compact operator for* $0 \leq s < t \leq T_0$.

In order to construct an impulsive evolution operator and investigate its properties.
First consider the following Cauchy problem :

$$\begin{cases} \dot{x}(t) & = A(t)x(t), \qquad t \in [0, T_0], \quad t \neq \tau_k, \\ \Delta x(\tau_k) & = B_k x(\tau_k), \qquad t = \tau_k, \quad k = 1, 2, \cdots, \sigma \\ x(0) & = x_0. \end{cases} \tag{2.3}$$

For every $x_0 \in X$, $\mathcal{D}$ is an invariant subspace of B_k, using Lemma 2.1 , step by step one can verify that the Cauchy problem (2.3) has a unique classical solution $x \in PC^1([0,T_0], X)$ represented by $x(t) = \mathcal{S}(t,0)x_0$, where $\mathcal{S}(.,.) : \triangle \to X$ given by

$$\mathcal{S}(t,s) = \begin{cases} U(t,s), & \tau_{k-1} \leq s \leq t \leq \tau_k, \\ U(t,\tau_k^+)(I+B_k)U(\tau_k, s), & \tau_{k-1} \leq s < \tau_k < t \leq \tau_{k+1}, \\ U(t,\tau_k^+)\left[\prod_{s<\tau_j<t}(I+B_j)U(\tau_j, \tau_{j-1}^+)\right](I+B_i)U(\tau_i, s), & \\ \qquad \tau_{i-1} \leq s < \tau_i \leq \ldots \leq \tau_k < t \leq \tau_{k+1}. \end{cases} \tag{2.4}$$

The operator $\mathcal{S}(t,s)$ $\big((t,s) \in \triangle\big)$ is called *impulsive evolution operator.*

Lemma 2.3. (see [6], page 5). *Let assumption (A1.1) , (A1.2) , (A2) and (A3) hold. The impulsive evolution operator $\mathcal{S}(t,s)$ has the following properties :*

1. $\mathcal{S}(t,s) \in \mathcal{L}(X)$, *for* $0 \le s \le t \le T_0$;
2. $\mathcal{S}(t+T_0, s+T_0) = \mathcal{S}(t,s)$, *for* $0 \le s \le t \le T_0$;
3. $\mathcal{S}(t+T_0, 0) = \mathcal{S}(t,0)\mathcal{S}(T_0,0)$, *for* $0 \le t \le T_0$;
4. $\mathcal{S}(t,s)$ *is compact operator , for* $0 \le s < t \le T_0$.

Definition 2.4. *For every $x_0 \in X$ and $f \in L^1\big([0,\infty), X\big)$, the function $x \in PC\big([0,\infty), X\big)$ given by*

$$x(t) = \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)f(s,x(s))ds \tag{2.5}$$

for all $t \in [0,T_0]$, is said to be a mild solution of system (2.1).

Definition 2.5. *A function $x \in PC([0,\infty), X)$ is said to be a periodic mild solution of system (2.1) if it is a mild solution and there exists $T_0 > 0$ such that $x(t+T_0) = x(t)$ for all $t \ge 0$.*

Definition 2.6. *A function $x \in PC([0,\infty), X)$ is said to be a T_0-periodic mild solution of system (2.1) if it is a mild solution and $x(t+T_0) = x(t)$ for all $t \ge 0$.*

Consider the following impulsive system ,

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + f(t,x(t)), \qquad t \in [0,T_0], \quad t \ne \tau_k, \\ \Delta x(\tau_k) &= B_k x(\tau_k), \qquad t = \tau_k, \quad k = 1,2,\cdots,\sigma \\ x(0) &= x_0, \end{cases} \tag{2.6}$$

where $\{A(t), t \in [0,T_0]\}$ is a family of closed densely defined linear unbounded operator on X, and $f : [0,\infty) \times X \to X$. A function $x \in PC([0,T_0], X)$ is called a mild solution of system (2.6) if x is given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s,x(s))ds.$$

Theorem 2.7. *If assumption (A1) hold, then system (2.6) has a unique mild solution $x \in C([0,T_0], X)$.*

Proof. Firstly, we consider the following general differential equation without impulse

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + f(t,x(t)), \qquad t > 0 \\ x(0) &= x_0, \end{cases} \tag{2.7}$$

Define a closed ball

$$\mathcal{B}(x_0,1) = \{x \in C([0,T_1], X) | \|x(t) - x(0)\|_X < 1, 0 \le t \le T_1\}$$

where T_1 will be chosen later. Define a map Q on $\mathcal{B}(x_0,1)$ by

$$(Qx)(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s,x(s))ds$$

and let $M = \sup_{t\in[0,T_0]} \|U(t,s)\|_{\mathcal{L}(X)}$.
Using assumption (A1.3), one can verify that $Q : \mathcal{B}(x_0,1) \to \mathcal{B}(x_0,1)$.
We have

$$\begin{aligned} \|(Qx)(t) - x_0\|_X &\le \|U(t,0)x_0 - x_0\|_X + \int_0^t \|U(t,s)\|_{\mathcal{L}(X)} \|f(s,x(s))\|_X ds \\ &\le \|U(t,0)x_0 - x_0\|_X + MK_1(\rho)t. \end{aligned}$$

Since $U(t,s)$ is the strongly continuous, there exists $\tau' > 0$ such that

$$\|U(t,0)x_0 - x_0\|_X \le \frac{1}{2}, \qquad \text{for all } \ t \in [0,\tau'].$$

Now, let $0 < \tau'' < \frac{1}{2MK_1(\rho)}$. Set $T_1' = min\{\tau', \tau''\}$, we have

$$\|(Qx)(t) - x_0\|_X \le 1, \qquad \text{for all } \ t \in [0,T_1'].$$

This mean that $(Qx)(t) \in \mathcal{B}(x_0,1)$. Hence, $Q : \mathcal{B}(x_0,1) \to \mathcal{B}(x_0,1)$.
Let $x, y \in \mathcal{B}(x_0,1)$. Using assumption (A1.3), we have

$$\begin{aligned} \|(Qx)(t) - (Qy)(t)\|_X &\le \int_0^t \|U(t,s)\|_{\mathcal{L}(X)} \|f(s,x(s)) - f(s,y(s))\|_X ds \\ &\le MK_2(\rho)t\|x - y\|_X. \end{aligned}$$

Now, let $0 < T_1'' < \frac{1}{2MK_2(\rho)}$, then

$$\|(Qx)(t) - (Qy)(t)\|_X \leq \frac{1}{2}\|x - y\|_X.$$

This means that the map Q is contraction map. We shall choose $T_1 = min\{T_1', T_1''\}$ such that Q is a contraction map on $\mathcal{B}(x_0, 1)$. By contraction map principle, there exists a unique fixed point, this implies that (2.7) has a unique mild solution on $[0, T_1]$.
Suppose $x(.)$ is a mild solution of (2.7), then we have

$$\begin{aligned}\|x(t)\|_X &\leq \|U(t,0)\|_{\mathcal{L}(X)}\|x_0\|_X + \int_0^t \|U(t,s)\|_{\mathcal{L}(X)}\|f(s,x(s))\|_X ds \\ &\leq M\|x_0\|_X + MK_1(\rho)t.\end{aligned}$$

By Gronwall inequality, we have obtain

$$\|x(t)\|_X \leq M\|x_0\|_X + MK_1(\rho)t \equiv \bar{M}.$$

That is, there exists a constant $\bar{M} = M\|x_0\|_X + MK_1(\rho)t > 0$ such that $\|x(t)\|_X \leq \bar{M}$ for all $t \in [0, T_0]$. Then we can prove the global existence of the mild solution of system (2.7) on $[0, T_0]$. □

Theorem 2.8. *If assumptions (A1) hold, then system (2.1) has a unique mild solution* $x \in PC([0, T_0], X)$.

Proof. For $t \in [0, \tau_1]$, Theorem 2.7 implies that system

$$\dot{x}(t) = A(t)x(t) + f(t, x(t)), \quad 0 < t \leq \tau_1, \quad x(0) = x_0,$$

has a mild solution on $I_1 = [0, \tau_1]$ which satisfies

$$x_1(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s, x_1(s))ds, \quad t \in [0, \tau_1].$$

Now, define

$$x_1(\tau_1) = U(\tau_1, 0)x_0 + \int_0^{\tau_1} U(\tau_1, s)f(s, x_1(s))ds,$$

so that $x_1(\cdot)$ is left continuous at τ_1. Next, on $I_2 = (\tau_1, \tau_2]$, consider system

$$\dot{x}(t) = A(t)x(t) + f(t, x(t)), \quad \tau_1 < t < \tau_2, \quad x_1(\tau_1^+) = (I + B_1)x_1(\tau_1).$$

Since $x_1 \in X$, we can use Theorem 2.7 again to get a mild solution on $(\tau_1, \tau_2]$ which satisfying

$$x_2(t) = U(t, \tau_1)x_1(\tau_1^+) + \int_{\tau_1}^t U(t,s)f(s, x_2(s))ds.$$

Now, define $x_2(\tau_2)$ accordingly so that $x_2(\cdot)$ is left continuous at τ_2. It is easy to see that Theorem 2.7 can be applied to interval $(\tau_1, \tau_2]$ to verify that $x_2(\tau_2) \in X$. Repeat the procedure above, use step-by-step approach on intervals $I_k = (\tau_{k-1}, \tau_k]$, $k = 3, 4, \ldots, \sigma$ $(\tau_\sigma = T_0)$ to get a mild solutions

$$x_k(t) = U(t, \tau_{k-1})x_{k-1}(\tau_{k-1}^+) + \int_{\tau_{k-1}}^t U(t,s)f(s, x_k(s))ds.$$

for $t \in (\tau_{k-}, \tau_k]$ and define $x_k(\tau_k)$ accordingly with $x_k(\cdot)$ left continuous at τ_k and $x_k(\tau_k) \in X$, $k = 1, 2, \ldots, \sigma$. Thus we obtain $x \in PC([0, T_0], X)$ is a mild solution of system (2.1) and given by

$$x(t) = \begin{cases} x_1(t), & 0 \leq t \leq \tau_1, \\ x_k(t), & \tau_{k-1} < t \leq \tau_k, \ k = 2, 3, \ldots, \sigma. \end{cases}$$

Next, by mathematical induction we can show that (2.5) is satisfied on $[0, T_0]$. First, (2.5) is satisfied on $[0, \tau_1]$. If (2.5) is satisfied on $(\tau_{k-1}, \tau_k]$, then for $t \in (\tau_k, \tau_{k+1}]$,

$$\begin{aligned} x(t) = x_{k+1}(t) &= U(t, \tau_k)x_k(\tau_k^+) + \int_{\tau_k}^t U(t,s)f(s, x_{k+1}(s))ds \\ &= U(t, \tau_k)(I + B_k)\Big[U(\tau_k, 0)x_0 + \int_0^{\tau_k} U(\tau_k, s)f(s, x(s))ds\Big] + \int_{\tau_k}^t U(t,s)f(s, x_{k+1}(s))ds \\ &= U(t,0)x_0 + \int_0^t U(t,s)f(s, x(s))ds. \end{aligned}$$

Thus (2.5) is also true on $(\tau_k, \tau_{k+1}]$. Therefore (2.5) is true on $[0, T_0]$. Next, we want to show that a mild solution is unique on $PC([0, T_0], X)$. Suppose that x, y are mild solutions of system (2.1) on $PC([0, T_0], X)$. Then by Theorem 2.7, we have

$$\|x(t) - y(t)\|_X \leq \int_0^t \|U(t,s)\|_{\mathcal{L}(X)} \|f(s, x(s)) - f(s, y(s))\|_X ds \leq MK_2(\rho) \int_0^t \|x(s) - y(s)\|_X ds.$$

It follows from Gronwall Lemma, we obtain $\|x(t) - y(t)\| = 0$ for all $t \in [0, T_0]$. That is, $x = y$. Therefore, system (2.1) has a unique mild solution. This completes the proof. □

To be able to apply the method in Pazy [1]. we also need the following lemma.

Lemma 2.9. ([1]). *Consider the nonhomogeneous initial value problem*

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, y(t)), & t > 0 \\ x(0) = x_0. \end{cases} \tag{2.8}$$

If $f \in L^1([0, \infty), X)$, then for every $x_0 \in X$ the initial value problem (2.8) has a unique solution which satisfies

$$x(t) = U(t, 0)x_0 + \int_0^t U(t,s)f(s, x(s))ds, \qquad 0 \leq t \leq T_0.$$

We consider the following system,

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, x(t)), & t \geq 0 \\ x(0) = x_0, \end{cases} \tag{2.9}$$

and we suppose that it has a global mild solution $x(t)$.

We also consider the following system,

$$\begin{cases} \dot{y}(t) = A(t)y(t) + f(t, x(t)), & t \geq 0 \\ y(0) = x(0). \end{cases} \tag{2.10}$$

By Lemma (2.9), system (2.10) has a unique mild solution $y(t)$.
Let $P : X \to X$ be the Poincar mapping, defined by

$$Px = y(T_0) = U(T_0, 0)x_0 + \int_0^{T_0} U(t,s)f(s, x(s))ds.$$

Finally, we consider the following system,

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, x(t)), & t \geq 0 \\ x(0) = Px, \end{cases} \tag{2.11}$$

which by Lemma (2.9) also has a unique mild solution $x(t)$.

We are now in a position to state and prove the basic tool for the proof existence of periodic mild solution.

Theorem 2.10. *System (2.9) has a T_0-periodic mild solution if and only if the mapping P has a fixed point.*

Proof. Let x be a T_0-periodic mild solution of system (2.9). Then x is clearly a T_0-periodic mild solution of system (2.10). Since x is T_0-periodic mild solution, $x(0) = x(T_0)$. Therefore $x(0) = x(T_0) = Px$, where x satisfy (2.11) and so $Px = x$. Conversely, let x be a fixed point of P. By definition, x satisfies (2.10) and since $x(0) = y(0)$. By Lemma (2.9) , show that $x(t) \equiv y(t)$ and hence $x(T_0) = y(T_0)$. Since $Px = x$, it follows from (2.11) that $x(0) = Px = y(T_0) = x(T_0)$. That is, $x(0) = x(T_0)$. The function $\psi(t) := x(t + T_0)$ is also a mild solution of (2.9). Since f is T_0-periodic, $\dot{\psi}(t) = \dot{x}(t + T_0) = A(t+T_0)x(t+T_0) + f(t+T_0, x(t+T_0)) = A(t)\psi(t) + f(t, \psi(t))$. Therefore $x(t) = x(t+T_0)$ for all $t \geq 0$. i.e., system (2.9) has a T_0-periodic mild solution. This completes the proof. □

3 Nonlinear Impulsive Periodic Systems With Parameter Perturbations

We consider the nonlinear impulsive periodic system with parameter perturbations as the following

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, x(t)) + p(t, x(t), \xi), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t) + c_k + q_k(x(t), \xi), & t = \tau_k, \end{cases} \tag{3.1}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$. In addition to assumptions (A1), we introduce the following assumption

Assumption (A4) ;

(A4.1) $c_k \in X$ and $c_{k+\sigma} = c_k$ for all $k \in \mathbb{N}$.

(A4.2) The Fréchet derivative $\frac{\partial}{\partial x} f(t,x)$ exists in $[0,\infty) \times X$. For each $y \in X$, $t \mapsto \frac{\partial}{\partial x} f(t,x)y$ is strongly measurable, $x \mapsto \frac{\partial}{\partial x} f(t,x)y$ is continuous. For every $\rho > 0$, there exists a constant $K_3(\rho) > 0$ such that

$$\left\| \frac{\partial}{\partial x} f(t,x) \right\|_{\mathcal{L}(X)} \leq K_3(\rho)$$

for all $t \geq 0$ and all $x \in X$ such that $\|x\|_X \leq \rho$.

(A4.3) $p : [0,\infty) \times S_\rho \times \Lambda \longrightarrow X$ is measuable for t such that $p(t+T_0, x, \xi) = p(t, x, \xi)$ and $q_k : S_\rho \times \Lambda \longrightarrow X$ such that $q_{k+\sigma}(x,\xi) = q_k(x,\xi)$, where $\Lambda \equiv (-\widetilde{\xi}, \widetilde{\xi}), (\widetilde{\xi} > 0)$ and $S_\rho = \{x \in PC\big([0,\infty), X\big) | \|x\|_{PC} < \rho\}$ and there exists a nonnegative function ω such that

$$\lim_{\xi \to 0} \omega(\xi) = \omega(0) = 0$$

and for all $t \geq 0, x, y \in S_\rho$ and $\xi \in \Lambda$ such that $\|p(t,x,\xi) - p(t,y,\xi)\|_X \leq \omega(\xi)\|x-y\|_X$ and $\|q_k(x,\xi) - q_k(y,\xi)\|_X \leq \omega(\xi)\|x-y\|_X$.

(A4.4) The Fréchet derivative $\frac{\partial}{\partial x} B_k(x)$ exists in X. For every $\rho > 0$, there exists a constant $\bar{h}_k(\rho) > 0$ such that

$$\left\| \frac{\partial}{\partial x} B_k(x) \right\|_{\mathcal{L}(X)} \leq \bar{h}_k(\rho)$$

for all $t \geq 0$, $k \in \mathbb{N}$ and all $x \in X$ such that $\|x\|_X \leq \rho$.

Definition 3.1. *A function $x \in PC([0,\infty), X)$ is said to be a mild solution of impulsive system (3.1) with initial condition $x(0) = x_0 \in X$ if x is given by*

$$x(t) = \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)[f(s,x(s)) + p(s,x(s),\xi)]ds + \sum_{0<\tau_k<t} \mathcal{S}(t,\tau_k)[c_k + q_k(x(\tau_k),\xi)]. \tag{3.2}$$

Definition 3.2. *A function $x \in PC([0,\infty), X)$ is said to be a periodic mild solution of system (3.1) if it is a mild solution and there exists $T_0 > 0$ such that $x(t+T_0) = x(t)$ for all $t \geq 0$.*

Definition 3.3. *A function $x \in PC([0,\infty), X)$ is said to be a T_0-periodic mild solution of system (3.1) if it is a mild solution and $x(t+T_0) = x(t)$ for all $t \geq 0$.*

First, we consider the following reference system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, x(t)), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t), & t = \tau_k, \end{cases} \tag{3.3}$$

and assume that $x_{T_0}(t)$ is a T_0- periodic mild solution of the reference system (3.3) which satisfies

$$x_{T_0}(t) = \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)f(s,x(s))ds. \tag{3.4}$$

Next, we consider the following variation system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + \frac{\partial}{\partial x} f(t, x_{T_0}(t))x(t), & t \neq \tau_k, \\ \Delta x(t) = \frac{\partial}{\partial x} B_k(x_{T_0}(t))x(t), & t = \tau_k, \end{cases} \tag{3.5}$$

and assume that the variation system (3.5) has only trivial solution.

Theorem 3.4. *Let assumption (A1) and (A4) holds. Suppose* $x_{T_0}(t)$ *be a* T_0*-periodic mild solution of the reference system (3.3) satisfies*

$$\rho_0 = \sup_{t\in[0,T_0]} \|x_{T_0}(t)\|_X .$$

Assume that

1. *system (3.5) has only trivial solution,*
2. *let* $\widetilde{\xi} > 0$ *and* $\varepsilon_o \in (0, \rho - \rho_0)$ *such that* $\eta < 1$ *with*

$$\eta := M\Big([K_2(\varepsilon_0) + K_3(\varepsilon_0)]T_0 + [h_k(\varepsilon_0) + \bar{h}_k(\varepsilon_0)]\sigma + [T_0 + \sigma] \sup_{\xi\in[0,\tilde{\xi}]} \omega(\xi)\Big)$$

where

$$M = \sup_{0\le s\le t\le T_0} \|\mathcal{S}(t,s)\|_{\mathcal{L}(X)} \quad \text{and} \quad \bar{h}_k(\varepsilon_0) = \sup_{k\in\mathbb{N},\ \|y\|\le\varepsilon_0} \left\| \frac{\partial}{\partial x} B_k(x_{T_0}(\tau_k) + y(\tau_k)) \right\|_X ,$$

3. *the following inequality is valid*

$$\sup_{t\in[0,T_0],\ |\xi|\le\tilde{\xi}} \Bigg\| \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)[p(s, x_{T_0}(s), \xi)]ds + \sum_{0\le\tau_k<T_0} \mathcal{S}(t,\tau_k)[c_k + q_k(x_{T_0}(\tau_k), \xi)] \Bigg\|_X \le \varepsilon_0(1-\eta).$$

Then for any constant $\rho > \rho_0 > 0$*, there exists a sufficiently small* $\widetilde{\xi} > 0$ *such that for every fixed* $\xi \in [0, \widetilde{\xi}]$ *system (3.1) has a unique* T_0*-periodic mild solution* $x^{\xi}_{T_0}(t)$ *satisfying*

$$\|x^{\xi}_{T_0}(t) - x_{T_0}(t)\| < \varepsilon_0 \quad \text{for all} \quad t \ge 0 \tag{3.6}$$

and $$\lim_{\xi\to 0} x^{\xi}_{T_0}(t) = x_{T_0}(t) \qquad \text{uniformly on } t.$$

Proof. Let $x(t) = x_{T_0}(t) + y(t)$, then we can change system (3.1) into

$$\begin{aligned} \dot{y}(t) &= A(t)y(t) + \frac{\partial}{\partial x} f(t, x_{T_0}(t))y(t) + o(t, y(t)) + p(t, x_{T_0}(t) + y(t), \xi), \ t \ne \tau_k, \\ \Delta y(t) &= \frac{\partial}{\partial x} B_k(x_{T_0}(t))y(t) + o_k(y(t)) + c_k + q_k(x_{T_0}(t) + y(t), \xi), \qquad t = \tau_k, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} o(t, y(t)) &= f(t, x_{T_0}(t) + y(t)) - f(t, x_{T_0}(t)) - \frac{\partial}{\partial x} f(t, x_{T_0}(t))y(t) \\ o_k(y(t)) &= B_k(x_{T_0}(t) + y(t)) - B_k(x_{T_0}(t)) - \frac{\partial}{\partial x} B_k(x_{T_0}(t))y(t) \end{aligned} \tag{3.8}$$

Let $PC_{T_0}([0,T_0]; X) := \{x \in PC([0,T_0]; X) \mid x(0) = x(T_0)\}$ with norm $\|x\|_{PC_{T_0}} = \sup_{t\in[0,T_0]} \|x(t)\|_X$.
Let us define

$$\mathcal{B} := \mathcal{B}(\varepsilon_0) = \{y \in PC_{T_0}([0,T_0]; X) \mid \|y\|_{PC_{T_0}} \le \varepsilon_0\} \tag{3.9}$$

and an operator $\Omega : \mathcal{B} \to PC_{T_0}([0,T_0]\,; X)$ such that

$$\begin{aligned} \Omega(x)(t) := \ & \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)\Big[o(s, y(s)) + p(s, x_{T_0}(s) + y(s), \xi)\Big]ds \\ & + \sum_{0\le\tau_k<t} \mathcal{S}(t,\tau_k)[o_k(y(\tau_k)) + c_k + q_k(x_{T_0}(\tau_k) + y(\tau_k), \xi)]. \end{aligned} \tag{3.10}$$

If $y \in \mathcal{B}$, then

$$\|x\|_{PC_{T_0}} = \|x_{T_0} + y\|_{PC_{T_0}} \le \|x_{T_0}\|_{PC_{T_0}} + \|y\|_{PC_{T_0}} \le \rho_0 + \varepsilon_0 \le \rho_0 + (\rho - \rho_0) = \rho.$$

From equation (3.10), we have

$$\Omega(x_{T_0})(t) = \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)\Big[p(s, x_{T_0}(s), \xi)\Big] + \sum_{0\le\tau_k<t} \mathcal{S}(t,\tau_k)[c_k + q_k(x_{T_0}(\tau_k), \xi)]. \tag{3.11}$$

For any $x, x_{T_0} \in \mathcal{B}$, then we have

$$\begin{aligned}
&\|\Omega(x) - \Omega(x_{T_0})\|_{PC_{T_0}} \\
&\quad \leq \int_0^t \|\mathcal{S}(t,s)\|_{\mathcal{L}(X)} \|o(s, y(s)) + p(s, x_{T_0}(s) + y(s), \xi) - p(s, x_{T_0}(s), \xi)\|_X \, ds \\
&\quad + \sum_{0 \leq \tau_k < t} \|\mathcal{S}(t, \tau_k)\|_{\mathcal{L}(X)} \|o_k(y(\tau_k)) + q_k(x_{T_0}(\tau_k) + y(\tau_k), \xi) - q_k(x_{T_0}(\tau_k), \xi)\|_X \\
&\leq \int_0^t \|\mathcal{S}(t,s)\|_{\mathcal{L}(X)} \Big(\|f(t, x_{T_0}(s) + y) - f(t, x_{T_0}(s))\| + \Big\| \frac{\partial}{\partial x} f(t, x_{T_0}(s)) y \Big\|_X \\
&+ \|p(s, x_{T_0}(s) + y(s), \xi) - p(s, x_{T_0}(s), \xi)\|_X \Big) ds \\
&+ \sum_{0 \leq \tau_k < t} \|\mathcal{S}(t, \tau_k)\|_{\mathcal{L}(X)} \Big(\|B_k(x_{T_0}(\tau_k) + y) - B_k(x_{T_0}(\tau_k))\| \\
&+ \Big\| \frac{\partial}{\partial x} B_k(x_{T_0}(\tau_k)) y \Big\|_X + \|q_k(x_{T_0}(\tau_k) + y(\tau_k), \xi) - q_k(x_{T_0}(\tau_k), \xi)\|_X \Big) \\
&\leq M \Big([K_2(\varepsilon_0) + K_3(\varepsilon_0) + \omega(\xi)] T_0 + [h_k(\varepsilon_0) + \bar{h}_k(\varepsilon_0) + \omega(\xi)] \sigma \Big) \|x - x_{T_0}\|_{PC_{T_0}}.
\end{aligned}$$

Let us choose $\widetilde{\xi} > 0$ and $\varepsilon_0 \in (0, \rho - \rho_0)$ such that $\eta < 1$ with

$$\eta := M \Big([K_2(\varepsilon_0) + K_3(\varepsilon_0)] T_0 + [h_k(\varepsilon_0) + \bar{h}_k(\varepsilon_0)] \sigma + [T_0 + \sigma] \sup_{\xi \in [0, \widetilde{\xi}]} \omega(\xi) \Big). \tag{3.12}$$

So
$$\|\Omega(x) - \Omega(x_{T_0})\|_{PC_{T_0}} \leq \eta \|x - x_{T_0}\|_{PC_{T_0}} \tag{3.13}$$

It follows from (3.11), (3.13) and assumption (3) that

$$\|\Omega(x)\|_{PC_{T_0}} \leq \|\Omega(x) - \Omega(x_{T_0})\|_{PC_{T_0}} + \|\Omega(x_{T_0})\|_{PC_{T_0}} \leq \eta \|x - x_{T_0}\|_{PC_{T_0}} + \varepsilon_0(1-\eta) \leq \eta\varepsilon_0 + \varepsilon_0(1-\eta) = \varepsilon_0$$

from which we know that $\Omega(x) \in \mathcal{B}$, then $\Omega : \mathcal{B} \to \mathcal{B}$ is a contraction mapping. Therefore, there exists a unique fixed point $y_1(t) \in \mathcal{B}$. From the fact that $y_1(t)$ is a solution of system (3.7), we know $x^{\xi}_{T_0}(t) = x_{T_0}(t) + y_1(t)$ is a T_0- periodic mild solution of (3.1) and satisfies

$$\|x^{\xi}_{T_0}(t) - x_{T_0}(t)\| = \|y_1(t)\| < \varepsilon_0.$$

So we have $\lim_{\xi \to 0} x^{\xi}_{T_0} = x_{T_0}(t)$ uniformly on t.

This completes the proof. □

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