การรวมเทคนิคถังความน่าจะเป็นและเทคนิคการขยายสหสัมพันธ์ ในการประเมินค่าตราสารหนี้ CDO

Probability Bucketing for Correlation Expansion in CDO Pricing

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บทคัดย่อ

เราพิจารณาที่จะรวมเอาสองวิธีการซึ่งก็คือ วิธีการขยายสหสัมพันธ์ วิธีการนี้จะแสดงให้เห็นว่าการประเมินค่าตราสารหนี้ CDO ซึ่ง obligors ของมันขึ้นแก่กันและกันนั้น สามารถที่จะอธิบายได้ในรูปของอนุกรมของราคา ซึ่งอยู่ในรูปแบบของ obligors ที่เป็นอิสระต่อกัน ดังนั้นจึงทำให้ความสลับซับซ้อนในการประเมินค่าตราสารหนี้ CDO และอนุพันธ์อื่นๆ ลดน้อยลง อีกวิธีการหนึ่งคือ วิธีการถังความน่าจะเป็น วิธีการนี้เป็นวิธีการที่ใช้ในการสร้างการแจกแจงค่าความเสียหาย (Loss distribution) วัตถุประสงค์ของเราคือต้องการที่จะพัฒนาวิธีการที่ใช้ในการประเมินค่าชั้นตราสารหนี้ CDO และอนุพันธ์อื่นๆ ที่มีคุณลักษณะไม่สลับซับซ้อนในการคำนวณ และปรับปรุงเวลาที่ใช้ในการคำนวณให้เร็วมากขึ้น เรายังสนใจรูปแบบในการสร้างการแจกแจงค่าความเสียหาย ซึ่งเป็นขั้นตอนสำคัญขั้นตอนหนึ่งในการประเมินค่าอนุพันธ์ เราพบว่าการนำวิธีการสองวิธีการนี้มารวมกันนั้นให้ค่าคำตอบที่แม่นยำเทียบเท่าคำตอบที่ได้จากการจำลอง Monte Carlo อย่างไรก็ตามในการจัดแจงช่วงของถังความน่าจะเป็นนั้นต้องมีแบบแผนในการจัดการ

คำสำคัญ : CDO ถังความน่าจะเป็น การขยายสหสัมพันธ์ การแจกแจงค่าความเสียหาย การประเมินค่า

ABSTRACT

We consider about to combine two techniques which are Correlation Expansion technique. This technique shows that CDOs, whose obligors are dependent, can be expressed as a series of prices in independent obligor model. So, this is much less complicated to price CDOs and credit basket derivatives. And another is Probability Bucketing technique which is the technique that uses to create loss distributions. Our purpose is to develop a method of approximating CDOs tranche price and credit basket derivatives that less complicated in the computation of the model's output and improve the calculation speed. We also focus on the way to contribute loss distributions which is one of the important parts for pricing credit derivatives. We find out that this combined method gives us the accurate value as Monte Carlo simulation. However, there is a procedure to set up the range of buckets.

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Keywords : Collateralize Debt Obligation, Probability Bucketing, Correlation Expansion, Loss distribution, pricing
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INTRODUCTION

As the credit derivatives market had grown over the past decade, products that depended on default correlations had become more popular such as a collateralized debt obligation (CDOs) and other credit basket derivatives. Though, the CDOs market was almost stopped since the end of 2007. We believe that CDOs market and other credit basket derivatives will return since there are several advantages of issuing credit risk securities to improve the liquidity or cash flow.

One of the major risks involved in the evaluation of credit derivatives is credit risk which is the distribution of financial loss caused by a broken financial agreement, for instance failure to pay interest or principal on a loan or bond. Approximating loss distribution that tells us the probabilities associated with different percentage losses in the portfolio is a part of the pricing process. The difficult task when we construct a loss distribution is the correlation among obligors since different correlation among obligors can make a loss distribution look completely different. In this paper, we presented the way to deal with this task through CDOs pricing framework.

Generally, a Monte Carlo Simulation, the market standard method, is applied to derive values for CDOs. However, a Monte Carlo Simulation is not very efficient for calculating price in real time because of the number of simulations that need to be run. This can take too much time for practitioners in times of strong market movements.

We will focus on the problem of pricing CDOs and credit basket derivatives. To determine joint default probabilities and develop a method of approximating CDOs tranche prices we combined two techniques:

1. The correlation expansion (P.Glasserman and S.Suchintabandid, 2007)
2. The probability bucketing approach (John Hull and Alan White, 2004)

For the first technique, the correlation expansion which is a method for approximating CDOs tranche prices by using a series of portfolios in which obligors are independent to approximate a portfolio whose obligors are correlated. The approximation is of the form:

$$E(L-y)^{+} \approx \sum_{J} const.\widetilde{E}_{J}(L-y)^{+}$$

where, for each label J, the expectation $\widetilde{E}_{J}(L-y)^{+}$ is the tranche price of a credit portfolio whose obligors are independent.

The advantages of this method are

- Pricing in independent obligor models are easy to compute.
- Since it less complicated to compute, it also takes less time compare with Monte Carlo simulation.

The disadvantage of this method is

- This technique require the very exact value of $\widetilde{E}_J(L-y)^+$.

The second technique is probability bucketing approach which calculates the probability distribution of the losses by time T, conditional on the values of the factors $Z_1, Z_2, ..., Z_d$ .They divide potential losses into ranges. The loss distribution is built up one obligor at a time. Andersen, L., J. Sidenius, and S. Basu [2003] also show the similar technique which is a special case of probability bucketing approach.

The advantages of this method are

- It gives us the advantage in term of computation speed.
- Give reasonably accurate answers compare with Monte Carlo simulation.
- Flexible to organize sizes of buckets.

The disadvantage of this method is

- It cannot deal with the correlation task.

So, we use probability bucketing technique to create independent total loss distributions and merge them with correlation expansion technique to help them create a dependent total loss distribution.

From two techniques above, we find out that using probability bucketing approach to estimate $\widetilde{E}_J(L-y)^+$ and substitute it in to

$$E(L-y)^+ \approx \sum_J const.\widetilde{E}_J(L-y)^+$$

which requires the exact value of $\widetilde{E}_J(L-y)^+$, it still give us the accurate value of $E(L-y)^+$ in condition of the size of the bucket.

PURPOSE

Our purpose of this work is to develop a method of approximating CDOs tranche price that less complicated in the computation of the model's output and improve the calculation speed.

METHODOLOGY

The main contribution of this paper is to combine two techniques. First, the Correlation Expansion method which is a method for approximating CDOs tranche prices by using a series of portfolios in which obligors are independent to approximate a portfolio whose obligors are correlated. The approximation is of the form:

$$E(L-y)^+ \approx \sum_J const.\widetilde{E}_J(L-y)^+ \quad (1)$$

The advantage of this method is pricing in independent obligor models are easy to compute. And the second method is the Probability Bucketing method which calculates the probability distribution of the losses by time T. The loss distribution is built up one obligor at a time. This approach has an advantage in term of computation speed and always gives reasonably accurate answers compare with Monte Carlo simulation.

To combine these two methods, first, we need to create perturbed probabilities ($\widetilde{p}_i^{(J)}$). Then, substitute the perturbed probabilities $\widetilde{p}_i^{(J)}$ to contribute the loss distribution by using probability bucketing method. The loss distribution tells us about the probabilities associated with different percentage losses in a portfolio. Consequently, we easily obtain expectations $\widetilde{E}_J(L-y)^+$ whose obligors are independent. But to make it be more realistic case, we need to consider about the correlation among obligors. Thus, we use the correlation expansion method to deal with this task. This method help us to obtain the expectation $E_t(L-y)^+$ whose obligors are dependent from expectations $\widetilde{E}_J(L-y)^+$ whose obligors are independent.

This combined method let us obtain an accurate value in such a short time compare to the traditional methods such as Monte Carlo simulation which is not very efficient for calculating price in real time because of the number of simulations that need to be run. We will show how this combined method work through an example.

Correlation Expansion

A numerical method for computing $E(L-y)^+$ was developed by P.Glasserman and S.Suchintabandid [2007]. This pricing method is based on computing joint probabilities of correlated normal random variables which is interpreted as the probability that obligors default at the same time.

$$P(X_1 > x_1, \ldots, X_M > x_M) \approx \sum_J const\,.\widetilde{p}_1^{(J)} \ldots \widetilde{p}_M^{(J)}$$

where X_i are correlated N(0,1) random variables, and x_i are real numbers.

So, the first step is to estimate the value of $\widetilde{p}_1^{(J)} \ldots \widetilde{p}_M^{(J)}$ which represent as the default joint probabilities of an independent obligor portfolio, each obligor has a marginal default probability of $\widetilde{p}_i^{(J)}$. The probability $\widetilde{p}_i^{(J)}$ can compute by using perturbation formula. Then, we can find the value of $\widetilde{E}_J(L-y)^+$.

The next step is to calculate the value of *const*, and substitute them into (1), therefore we receive $E(L-y)^+$.

In order to calculate $E(L-y)^+$ the method is explained below.

$$\begin{aligned} E(L-y)^+ &\approx \sum_J const\,.\widetilde{E}_J(L-y)^+ \\ &= \alpha_0 + \alpha_1 t + \alpha_2 \frac{t^2}{2!} + \ldots \end{aligned}$$

Then the coefficient α_n in the above expansion can be illustrated as follows

$$\sum_{J \in D^n} w_J \widetilde{E}_J (L - y)^+ \quad \xrightarrow{\theta \downarrow 0} \quad \alpha_n$$

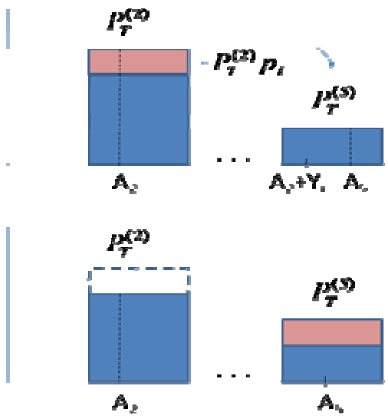
where the weight w_J is given as follows: $w_J := 1$ if n = 0; otherwise $w_J = \varpi_J / (2\theta^2)^n$. Define $\varpi_0 := -2(\varpi_1 + \ldots + \varpi_d)$ and $\varpi_J := \varpi_{|j_1|} \cdots \varpi_{|j_n|}$ for all $J = (j_1, \ldots, j_n)$ from the set D^n.

Probability Bucketing Approach

This method was developed by John Hull and Alan White [2004]. The purpose of this technique is to estimate the probability that the total loss lie in the k-th bucket for all k. Assume that the recovery rate is known and there are M obligors, then:

1. Divide potential losses into ranges: $\{0, b_0\}, \{b_0, b_1\}, \ldots, \{b_{K-1}, \infty\}$
2. Calculate the conditional probability that the loss by time T will be in the k-th bucket ($P_T^{(k)}$) and the mean loss conditional that the loss is in k-th bucket (A_k) by first assume that there are no obligors.
3. Built up one obligor at a time.

The only assumption in the iterative procedure is that we concentrate at the recent value of A_k for all the probability associated with bucket k. Suppose that we have calculated the $P_T^{(k)}$ and A_k when the first i-1 obligors are considered.



Example

Suppose we have a portfolio of 50 obligors. Each obligor has a default probability equal to 0.02. The given of the loss amount of obligor k is $C_k = k$. The perturbation parameter θ that we use is 0.1. And loading matrix A is a sparse matrix size 50 x 5. For the first column, the elements from a(1,1) to a(12,1) are equal to 0.2.

The next column, from a(9,2) to a(22,2) are 0.2, and elements a(19,3) to a(32,3) of the third column, a(29,4) to a(42,4) of the fourth column and a(39,5) to a(50,5) of the fifth column are equal to 0.2 as well. We want to calculate $E_t(L - y)^+$ at t = 1 for the given y = 50 , 75 , 100 , 200. We will compare the results among Monte Carlo simulation, the combined method with the interval of buckets equal to 1 unit (A), and the combined method whose intervals of buckets are in different ranges (B).

Case I: compare between Monte Carlo simulation and (A). Then the value of $E_t(L-y)^+$ at t = 1 is

Table 1

y	alpha 0	alpha 1	alpha 2	result	Monte Carlo
50	4.4497	0.3861	0.046	4.8588	4.7847 ± 0.04
75	1.5082	0.2818	0.0547	1.81735	1.773 ± 0.04
100	0.4352	0.1458	0.0473	0.60465	0.5888 ± 0.04
200	0.0011	0.0016	0.0022	0.0038	0.004 ± 0.04

Case II: compare between Monte Carlo simulation and (B). Then the value of $E_t(L-y)^+$ at t = 1 is

Table 2

y	alpha 0	alpha 1	alpha 2	result	Monte Carlo
50	4.4497	0.3861	0.0463	4.85895	4.7847 ± 0.04
75	1.5082	0.2818	0.055	1.8175	1.773 ± 0.04
100	0.4352	0.1458	0.0476	0.6048	0.5888 ± 0.04
200	0.0011	0.0016	0.0022	0.0038	0.004 ± 0.04

We find out that even we divide intervals to be 1 unit each bucket or we split them into different ranges they give us quite accurate value compare to Monte Carlo simulation with a simulation number of 1,000,000 times. And this method has an advantage in term of computing time. For Monte Carlo simulation, to receive the answer it takes around 2.5 hours for a simulation number of 1,000,000 times. While it take only about 152 seconds for (A) and 21 seconds for (B). This means, in case I, it was 57 times faster and 416 times faster in case II.

However, there is a procedure to set up the range of buckets. It is important to keep the range of early buckets to be narrow such as 1 unit each. And to consider which a specific number is that we can start to make buckets wider, we use the cumulative distribution function (cdf.) of obligors to help us in this decision.

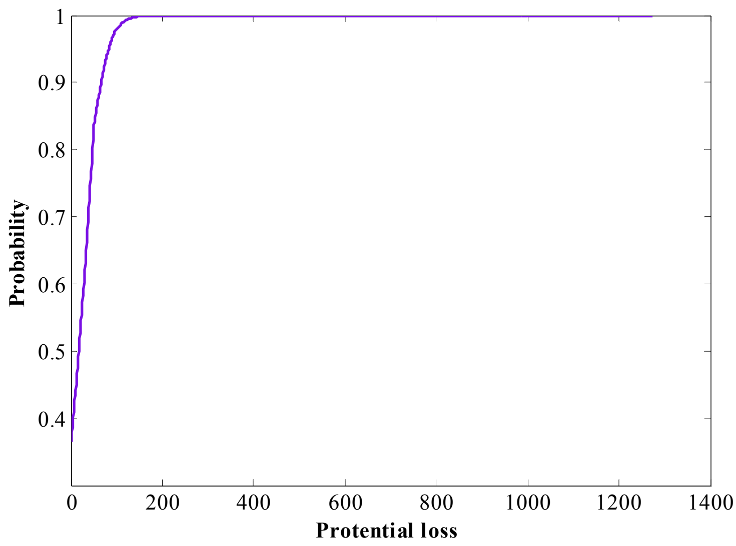


Figure 1 Cumulative distribution function

From the figure 1 we see that the graph sharply increase from the amount of potential loss of 0 to roughly 100. Then, gradually raise and stable. So, we can choose a specific number around 100 something. And divide buckets to be quite narrow from 0 to that number and make interval to be wider and wider for the rest of buckets. It also works for the range with the decimal. Nevertheless, the answer will be inaccurate if we create intervals of buckets too wide.

The running time of this combined method depends on the number of buckets and obligors. For this example which we have 50 obligors, the running time is about 120 seconds per 1,000 buckets.

RESULT

We find out that this combined method gives us the accurate value as Monte Carlo simulation to estimate the CDO tranche price at time *t*. And it is much less time consuming to calculate than Monte Carlo simulation. This method works in both high and low potential loss cases. Furthermore, no matter the default probabilities of each obligor are high or low, this method still give us the precise answer. However, the method is suitable just for portfolios that have low correlation. Also, to organize buckets is very important. To set buckets too wide, the speed of computation increase a lot, but the answers are not accurate. So, we need to arrange bucket sizes that make the calculation time fastest as possible but still give us the accurate value.

Since this combined method gives us the accurate answer and save time a lots, which is efficient to generate price in real time, it is also quite sensitive in the way of setting up buckets. So, we need to be careful when we set them up. And thing that help us to organize buckets easier is a cumulative distribution function. We use the probability bucketing method to create it and this take only about a few seconds to compute.

ANALYSIS

The speed of the combined method depends on number of obligors, number of buckets and how you write the program code. In this paper, we use the program call "MATLAB". The key that make the program (MATLAB) generate faster is the technique that we created sparse matrix of probabilities and updated probabilities in every bucket at the same time for every one obligor that added in, instead of using loops to update probabilities which can update only one bucket a loop. We also developed the program code to be more flexible in case that there is more than one probability that move to the same bucket.

CONCLUSION

We develop a method which improves the calculation speed and less complicated in term of computation to approximate the CDO tranche price. We combine two techniques; the correlation expansion (P.Glasserman and S.Suchintabandid, 2007) which give an advantage that pricing in independent obligor models are easy to compute and probability bucketing approach (John Hull and Alan White, 2004) which give the advantage in term of calculation speed. And find out that we can use this combined method which gives us the accurate value as Monte Carlo simulation to estimate the CDO tranche price at time *t*. However, we need to be careful when we organize buckets because it is quite sensitive. If we disorganize buckets, the result will be inaccurate.

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