การวิเคราะห์ความอ่อนไหวของแบบจำลองที่ให้ในการประเมินค่าตราสาร CDO
Sensitivity Analysis of CDO Pricing Models

รมณี บุญดีเจริญ[1]

Romani Boondicharern[1]

บทคัดย่อ

เราเปรียบเทียบแบบจำลองในการประเมินค่าตราสาร CDO และระดับการขึ้นแก่กันระหว่างความน่าจะเป็นของการไม่สามารถปฏิบัติตามสัญญาและความอ่อนไหวของระยะเวลาที่จะไม่สามารถปฏิบัติตามสัญญา โดยผ่านแบบจำลอง Gaussian, stochastic correlation student t และ Clayton Copula เราเปรียบเทียบความอ่อนไหวในแต่ละชั้นของแบบจำลองนี้ ต่ออัตราส่วนต่างของผลตอบแทนจากการจัดอันดับความน่าเชื่อถือของแต่ละคู่สัญญา obligors ซึ่งเป็นหลักทรัพย์การลงทุน เราจะทดสอบว่าแบบจำลองที่แตกต่างกันนั้นจะให้ค่าความอ่อนไหวของความน่าจะเป็นที่จะไม่สามารถปฏิบัติตามสัญญาแตกต่างกันหรือไม่ และยังศึกษาว่าแบบจำลองทุกแบบจะให้ผลว่าชั้นใดมีความอ่อนไหวต่อการเปลี่ยนแปลงของการจัดอันดับความน่าเชื่อถือของคู่สัญญา obligor เหมือนกันหรือไม่ มากกว่านั้นเรายังศึกษาความอ่อนไหวของราคาของชั้นซึ่งสัมพันธ์กับตัวแปรสหสัมพันธ์ เราต้องการสังเกตว่าแบบจำลองทุกแบบเห็นพ้องในความสัมพันธ์เหมือนกันหรือไม่ สุดท้ายเราต้องการศึกษาว่าแบบจำลองที่เหมาะสมดีกับตลาดแต่ละแบบนั้นมีรูปแบบของความอ่อนไหวเหมือนกันหรือไม่

คำสำคัญ : CDO จัดอันดับความน่าเชื่อถือ ตัวแปรสหสัมพันธ์ ชั้น ความอ่อนไหว

ABSTRACT

We compare some popular CDO pricing models and the degree of dependence between default probability and sensitivity to default times are modeled through Gaussian, stochastic correlation, Student t, and Clayton copulas. We compare these models which give less tranche sensitivity to the credit spread of obligors, n, the underlying portfolio. We will investigate whether different models give different probability default sensitivity and will see if all models agree while tranche is most sensitive to changes in the credit rating of obligors Moreover, we investigate sensitivity of tranche premium with respect to correlation parameter. We want to observe if all models agree on relationship as well. Lastly, we investigate whether models that fit market well share a sensitivity pattern.

Keywords : CDO, Credit spread, Correlation parameter, Tranche, Sensitivity

E-mail : n.nan86@gmail.com

[1] หลักสูตรวิทยาศาสตรมหาบัณฑิต สาขาวิชาการเงิน คณะพาณิชยศาสตร์และการบัญชี จุฬาลงกรณ์มหาวิทยาลัย กรุงเทพฯ 10330
Master of Science in Finance, Department of Banking and Finance, Faculty of Commerce and Accountancy, Chulalongkorn University, Bangkok 10330

INTRODUCTION

In this paper, we consider a collateralized debt obligation (CDO) and present a comparison of some popular CDO pricing model.

1. The Gaussian copula, the model is widely used by the financial industry, was introduced to the credit field by Li [2000].
2. A stochastic correlation extension of the Gaussian copula.
3. The Student *t* extension of the Gaussian copula with six and twelve degrees of freedom.
4. A double *t* one factor model
5. The Clayton copula model

We focus on "copula models" since there are mostly used in the credit derivatives markets, though the factor approach also applies to various intensity models (see Mortensen [2006] for an example). The pricing of synthetic CDOs involves the computation of aggregate loss distributions over different time horizons. In our "bottom-up" approach, CDO tranche premiums depend upon the individual credit risk of names in the underlying portfolio and the dependence structure between default times.

According to Burtschell, Gregory, and Laurent [2008], they compare some popular CDO pricing models. Dependence between default times is modeled through Gaussian, stochastic correlation, Student t, double t, Clayton and Marshall-Olkin copulas. They detail the model properties and compare the semi-analytic pricing approach with large portfolio approximation techniques. They have mentioned that base correlation is monotonic with respect to the model's dependence parameter. However, they did not compare Mezzanine, Senior, and relationship among tranches. They concluded that Clayton tends to give same tranche premium as Gaussian and cannot product implied correlation smile. Student T also cannot construct correlation smile. On the other hand, Double-T and stochastic correlation appear to fit the skew better. However, they did not link this to sensitivity analysis.

The contributions of this paper are

- Investigate tranche sensitivity to the credit rating of obligors, n the underlying portfolio. This issue finds practical implication when practitioners wish to hedge CDO tranche with other credit derivatives such as CDS. We will investigate whether different models give different probability default sensitivity and will see if all models agree while tranche is most sensitive to changes in the credit rating of obligors.
- Investigate sensitivity of tranche premium with respect to correlation parameter. His issue has important practical implication, since practitioner offer require calibrating the correlation parameter to fit the market quotes. While it is well known that equity monotonic with correlation parameter, no one has investigated mezzanine and senior tranche. We also want to see if all models agree on relationship

- Investigate whether models that fit market well share a sensitivity pattern. We hypothesize that the ability to fit market quotes can be determined by the pattern of tranche sensitivity to correlation parameter. Indeed, we shall see that Double-t and Stochastic correlation which have been shown to fit market well , Burtschell, Gregory, and Laurent [2008], share a correlation sensitivity pattern, while Clayton and Student-t do not have this pattern.

Financial crisis had become a current issue after the United States economy officially sank into a recession. One of the main reasons that cause the downturn in the United States economy arises from the underestimation of the credit default probability and correlation among obligors. Before the Hamburger crisis, financial industry was too optimistic about the ability to return of debtor. In addition, the analyst did not expect a high dependence on the default correlation and therefore it was assumed to be insignificant.

The paper is organized as follows: The first section, we talk about our contribution. Then, second section recalls the factor or conditional independence approach. This section also provided the basic understanding of each model. We consider Gaussian, Stochastic Correlation, Student T, Double T, and Clayton Copula. The third section introduced to the method which we consider on large portfolio approximation and perturb parameter. The fourth section described the payoff of CDOs. The fifth section considered on the result.

METROLOGY

In this study, we use the same numerical example with Burtschell, Gregory, and Laurent [2008] for comparability. For each models, we will perturb the parameters to see the difference in outcome it will give us. We will investigate the change in both default probability and correlation and find the error occur to the premium. Moreover, we will observe model that would give less sensitivity. In order to conduct the CDO tranche premium, we created MATLAB Code to simulate the outcome for each model using Monte- Carlo simulation. A sample of MATLAB Code is located in the Appendix.

In a CDO, default losses on the credit portfolio are divided along some thresholds or the attachment points and allocated to the different tranches. Our example consider on a three tranches CDO, denoted as equity, mezzanine, and senior, as shown in the Figure 1. Each tranche are divided by the attachment point which are A and B.

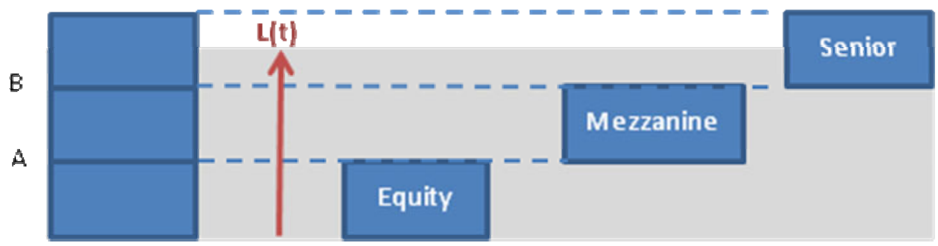

Figure 1 Three tranches of CDO

The cumulative default payment, which denote by L(t), can be separated into three scenarios. The first case is the case where cumulative default payment is less than attachment point A. In Figure 2, it shows the scenarios where the cumulative default payment in each tranches. In this scenario, equity tranche default payment, which can be refer as E(t), will be equal to L(t) and for both Mezzanine default payment, which indentify as M(t) and senior tranches default payment, which denote as S(t), are equal to zero.

In the second scenario, the cumulative default payment is less than attachment point B. For the equity tranche, default payments are denote by E(t)=A. For the mezzanine tranche, default payments are denote by M(t)=L(t)-A. In the senior tranche, default payment is equal to zero.

In the last case, the cumulative default payment more than attachment point B. In this scenario, default payment tranche is equal to A for equity and equal to B-A for Mezzanine tranche. For the senior tranche, default payment is equal to L(t)-B.

As mention that default payments are separated into three scenarios, it can be summarized of cumulative default payment of each tranche by these equations below.

$$E(t) = L(t)1_{L(t)<A} + A\,1_{L(t)>A} \tag{3.1}$$

$$M(t) = [L(t) - A]1_{A<L(t)<B} + [B - A]\,1_{L(t)>B} \tag{3.2}$$

$$S(t) = [L(t) - B]1_{L(t)>B} \tag{3.3}$$

The holder of synthetic CDO for the mezzanine tranche receives at time T a principal payment of $M(\infty) - M(T)$. $M(\infty)$ denoted the initial nominal of mezzanine tranche which is equal to $B - A$ and $M(\infty) - M(T)$ denotes the remaining nominal of the tranche

The payments are usually equal to a floating rate plus a fixed margin, which is specific to each tranche, and also based on the outstanding nominal on the tranche. For each payment dates, let set $t_i, i = 1, \dots, I$ which $t_I = T$.

The interest payment at t_i is equal to $\Delta_{i-1,t_i}(M(\infty)$ – M (t_i)) ($Libor_{t_{i-1}} + X)$ where X is the CDO margin, $\Delta_{i-1,i}$ represent the length of period, and $Libor_{t_{i-1}}$ is the Libor rate for the period. There are some accrued interest payments since the interest is based on outstanding nominal at the end of the period. In the case where name j defaults between t_{i-1} and t_i, the associated accrued interest payment is equal to $(\tau_j - t_{i-1})(Libor_{t_{i-1}} + X)(M(\tau_j) - M(\tau_j)^-)$.

The CDO tranche can be divided into a non defaultable amortizing floating rate note plus a default swap transaction where the CDO margin is exchanged against the default payment on the tranche. More exactly, there is a payment on the $(M(t)- M(t)^{-})$ at every jump of M(t). The CDO margin payments are equal to $X\,\Delta_{i-1,t_i}(M(\infty)$ – M $(t_i))$ at the payment dates plus some accrued payments which are $X(\tau_j - t_{i-1})(M(\tau_j)- M(\tau_j)^{-})$ on default date τ_j. As the above figure we will value separately the default and margin leg.

RESULT AND ANALYSIS

1. Sensitivity with respect to correlation

For the Sensitivity analysis with respect to correlation result, we found that all model agree in Equity and Senior Tranche. Equity coupon spread decreases with correlation. Nonetheless, senior premium increase with correlation. Most models agree in mezzanine which creates “bump”. However, student-t exhibit very irregular behavior in mezzanine. It generates “u-shape”. This might result in frown rather than smile correlation, as reported.

As suggested by, Burtschell, Gregory, and Laurent [2008], find that Gaussian, Clayton and Student-T cannot produce correlation smile, while Double-T and Stochastic Correlation can produce correlation smile. From plotting Mezzanine premium and senior premium against equity coupon spread, we detect a pattern that might be characteristic of smile-producing models. We found that, for double-t and Stochastic Correlation, the graph lies below Gaussian for mezzanine and above for senior. Clayton does not exhibit such pattern. For Student-T, it was very irregular. That might be the reason why it produces frown.

2. Sensitivity of CDO tranches with respect to Default Probability

<u>Sensitivity to Default Probability of Equity tranche</u>

We observe that Clayton copula is more sensitivity to default probability than Gaussian copula. Furthermore, Gaussian copula is more sensitivity to default probability than Student T copula. In addition, the student t with higher degree of freedom is conversing to Gaussian. For Double T and Stochastic Correlation Copula in equity tranche, there is not much difference. Therefore, those are inconclusive whether which is more sensitive. Interestingly, Double t has s-shape curves.

We observed that as the degree of t increase, it converse to Gaussian copula. The result of student-t come out just like what we expected since the difference between Gaussian and Student-t are the sample size. When we consider on delta-hedge, we can conclude that delta of Clayton is more than delta of Gaussian, and delta of Gaussian is more than delta of Student T since Clayton is more sensitive than Gaussian and Gaussian is more sensitive than student-t. As suggested by Burtschell, Gregory and Laurent (2008), the market premium gives “correlation smile”. If Double T fits market well, this means that for some range of spreads, Equity premium will change sharply relative to change in underlying speared.

Sensitivity to Default Probability of Mezzanine tranche

For the sensitivity to default probability of the mezzanine tranche, the credit spread in all models is close. Consequently, Mezzanine tranche is inconclusive. Again, double T copula is the most irregular than the other. It also gives S-shape in this tranche.

Sensitivity to Default Probability of senior tranche

In senior tranche, Gaussian is more sensitive than Clayton Copula. Student T is more sensitive than Gaussian. For student-t with higher degree of freedom premium is approaching Gaussian as well. We can conclude that model that gives more sensitivity to Equity tranche will be less sensitive for senior tranche. As a consequence, delta-hedge will be different base on different models.

Double-T and stochastic correlation are indistinguishable from Gaussian. Although double-t is most irregular. Since Double-T and stochastic correlation very similar to Gaussian in sensitivity, ability to produce implied correlation smile seems to depend on sensitivity to underlying correlation alone.

CONCLUSION

This paper is devoted to the sensitivity analysis to the default probability and the degree of dependence between defaults of various models. The models under investigation are one factor Gaussian copula, stochastic correlation extension of the Gaussian copula, Clayton copula, Student t copula, and double t copula.

- Compared to Gaussian all model agree in both equity and senior tranche. Equity premium decrease with correlation and senior premium increase with correlation. Almost every model agree in Mezzanine which create “bump”, but student-t create frown rather than “correlation smile”.
- Compare to Gaussian, Clayton and Student T produce very different default probability. Therefore, delta-hedging under different model will be different. Also model that gives more sensitivity to equity will give less sensitivity to senior and vice versa.
- Compared to Gaussian, stochastic correlation and double-T not very different. Double T is the most irregular compared to other models. Interestingly, stochastic correlation and double-T can produce implied correlation smile while Gaussian cannot. It seems, therefore, that ability to fit market quote lies in interaction with pool’s correlation, not default probability. In other words, hedging and fitting market seem to be separate problems.

REFERENCE

[1] Andersen, L., J. Sidenius and S. Basu, 2003, *All your Hedges in One Basket*, RISK, November, 67-72.

[2] Hull, J., and A. White, 2004, *Valuation of a CDO and an nth to Default CDS without Monte Carlo Simulation*, Journal of Derivatives, 2, 8-23.

[3] Laurent, J.-P., and J. Gregory, 2005, *Basket Default Swaps, CDOs and Factor Copulas*, Journal of Risk, Vol. 7, n°4, 103-122.

[4] X. Burtschell, J. Gregory and J.-P. Laurent, 2008, *A comparative analysis of CDO pricing models*, April

[5] Gregory, J., and J.-P. Laurent, 2004, *In the Core of Correlation*, RISK, October, 87-91.

[6] Li, D., 2000, *On Default Correlation: a Copula Approach*, Journal of Fixed Income, 9, 43-54.